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Challenges in Semileptonic B decays
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# Heavy-quark masses and HQE matrix elements from Lattice QCD

in collaboration with: Paolo Gambino (Torino) and Aurora Melis (Valencia)

## outline

- the extraction of the CKM element $V_{cb}$ from the OPE analysis of inclusive semileptonic B-meson decay rates requires the precise knowledge of c-quark and b-quark masses and of the matrix elements of some dimension-five and dimension-six operators
- state-of-art of the lattice determinations of charm and bottom quark masses (FLAG based)
- the aim of the talk is to present recent (unquenched) **lattice** determinations of the matrix elements of **dimension-four and dimension-five operators** obtained by studying the HQE of pseudoscalar and vector heavy-light meson masses (an estimate of the matrix elements of dimension-six operators will be also presented)

Gambino, Melis, SS: PRD 96 (2017) 014511 [1704.06105]
based on ETMC $N_f = 2 + 1 + 1$ configurations

FNAL/MILC/TUMQCD: 1802.04248
based on PRD 97 (2018) 0340503 and
MILC $N_f = 2 + 1 + 1$ configurations

# motivations

*** extraction of $V_{cb}$: exclusive versus inclusive determinations**

- exclusive semileptonic decays + LQCD@w=1 (FNAL) + shape parameterizations

$$\left|V_{cb}\right| = \left(39.0 \pm 0.8\right)\cdot 10^{-3} \quad \left(\mathrm{B} \to \mathrm{D}^{*}\ell\nu,\ \mathrm{LQCD},\ \mathrm{CLN}\right)$$

$$= \left(39.2 \pm 1.0\right)\cdot 10^{-3} \quad \left(\mathrm{B} \to \mathrm{D}\ell\nu,\ \mathrm{LQCD},\ \mathrm{CLN}\right)$$

PDG ’17, HFLAV ’16

$$= \left(41.9^{+2.0}_{-1.9}\right)\cdot 10^{-3} \quad \left(\mathrm{B} \to \mathrm{D}^{*}\ell\nu,\ \mathrm{LQCD},\ \mathrm{BGL}\right)$$

Grinstein, Kobach ’17
Bigi, Gambino, Schacht ’17

$$= \left(40.5 \pm 1.0\right)\cdot 10^{-3} \quad \left(\mathrm{B} \to \mathrm{D}\ell\nu,\ \mathrm{LQCD},\ \mathrm{BGL/BCL/CLN}\right)$$

Bigi, Gambino ’16

- inclusive semileptonic decays + OPE + HQE parameters from experimental data

$$\left|V_{cb}\right| = \left(42.11 \pm 0.74\right)\cdot 10^{-3}$$

Gambino, Healey, Turczyk ’16

*** improving of the precision and clarifications of open issues:**

- (dis)favours new physics interpretations (in connection with new physics effects in rare decays)

- is interesting in view of the *R(D)* and *R(D*)* anomalies

$$R(D)^{SM} = 0.299(3) \qquad R(D)^{\exp} = 0.407(46)$$

$$R(D^{*})^{SM} = 0.252(3) \qquad R(D^{*})^{\exp} = 0.304(15)$$

PDG ’17

# inclusive semileptonic decay rate

*** OPE → Heavy Quark Expansion (HQE)** [see Benson et al. ’03]

$\mu_{soft} \sim 1\ GeV$

$$\Gamma_{sl} = \Gamma_{tree} \left\{ 1 + a^{(1)} \frac{\alpha_s}{\pi} + a^{(2)} \left( \frac{\alpha_s}{\pi} \right)^2 + \frac{\mu_\pi^2\big|_B}{m_b^2} \left( -\frac{1}{2} + b^{(1)} \frac{\alpha_s}{\pi} \right) + \frac{\mu_G^2\big|_B}{m_b^2} \left( c^{(0)} + c^{(1)} \frac{\alpha_s}{\pi} \right) \right.$$

$$\left. + \frac{\rho_D^3\big|_B}{m_b^3} \left( d^{(0)} + d^{(1)} \frac{\alpha_s}{\pi} \right) + \frac{\rho_{LS}^3\big|_B}{m_b^3} \left( e^{(0)} + e^{(1)} \frac{\alpha_s}{\pi} \right) + O\left( \frac{1}{m_b^4} \right) \right\}$$

$$\Gamma_{tree} = \left| V_{cb} \right|^2 \frac{G_F^2 m_b^5}{192\pi^3} \eta_{ew} \left( 1 - 8r + 8r^3 - r^4 - 12r^2 \ln r \right) \qquad r = \frac{m_c^2}{m_b^2}$$

$a^{(1,2)}, b^{(1)}, c^{(0,1)}, d^{(0,1)}, e^{(0,1)}$ are functions of $r$

* the main ingredients are the charm and bottom quark masses and the HQE matrix elements of dimension-5 and dimension-6 operators in the B-meson

$$\mu_\pi^2\big|_B \equiv \frac{1}{2M_B} \left\langle B \middle| \bar{b} \left( i\vec{D} \right)^2 b \middle| B \right\rangle \quad (kinetic) \qquad \mu_G^2\big|_B \equiv \frac{1}{2M_B} \left\langle B \middle| \bar{b} \left( \frac{i}{2} \sigma_{\mu\nu} G^{\mu\nu} \right) b \middle| B \right\rangle \quad (chromomagnetic)$$

$$\rho_D^3\big|_B \equiv \frac{1}{2M_B} \left\langle B \middle| \bar{b} \left( -\frac{1}{2} \vec{D} \cdot \vec{E} \right) b \middle| B \right\rangle \quad (Darwin) \qquad \rho_{LS}^3\big|_B \equiv \frac{1}{2M_B} \left\langle B \middle| \bar{b} \left( \vec{\sigma} \cdot \vec{E} \times i\vec{D} \right) b \middle| B \right\rangle \quad (convection)$$

## fits of experimental semileptonic moments

$m_b^{kin}(1\,GeV) = 4.546(21)\,GeV$

$\mu_\pi^2\big|_B = 0.432(68)\,GeV^2$

$\mu_G^2\big|_B = 0.355(60)\,GeV^2 \quad (from\ B^* - B\ splitting)$

$m_c^{\overline{MS}}(3\,GeV) = 0.987(13)\,GeV$

$\rho_D^3\big|_B = 0.145(61)\,GeV^3$

$\rho_{LS}^3\big|_B = -0.17(10)\,GeV^3 \quad (from\ HQ - SR)$

Gambino et al. ’16
(kinetic mass scheme)

# charm quark mass from lattice QCD

## FLAG-3 review (Nov. ’15)

$$\bar{m}_c(3\,GeV) = 0.996(25)\,GeV \qquad N_f = 2+1+1 \qquad (\sim 2.5\%)$$

$$= 0.986(6)\,GeV \qquad N_f = 2+1 \qquad (\sim 0.6\%)$$

$$= 1.03(4)\,GeV \qquad N_f = 2 \qquad (\sim 4\%)$$

## updates / new results

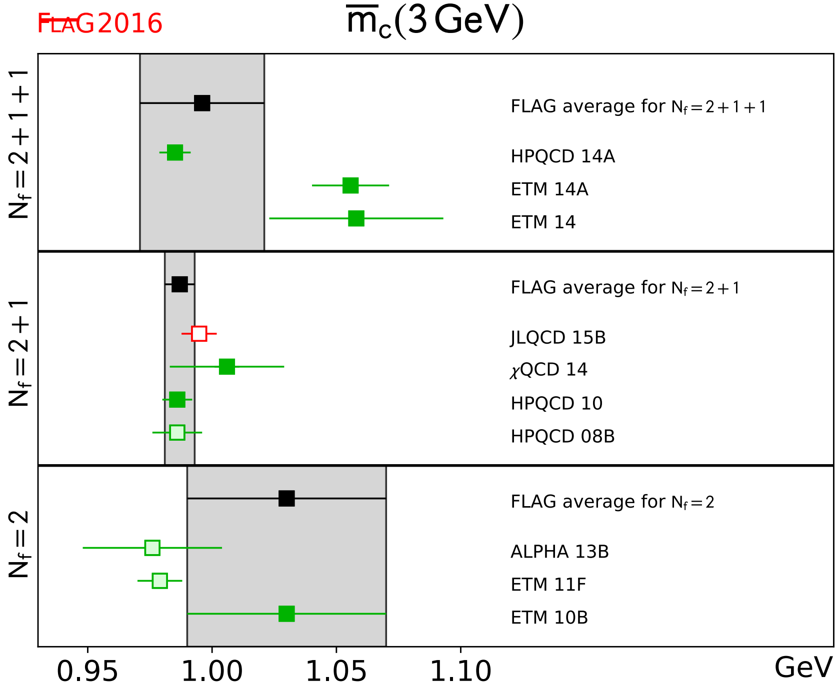
$$\bar{m}_c(3\,GeV) = 0.9843(55)\,GeV \qquad N_f = 2+1+1$$ [FNAL/MILC/TUMQCD 18]

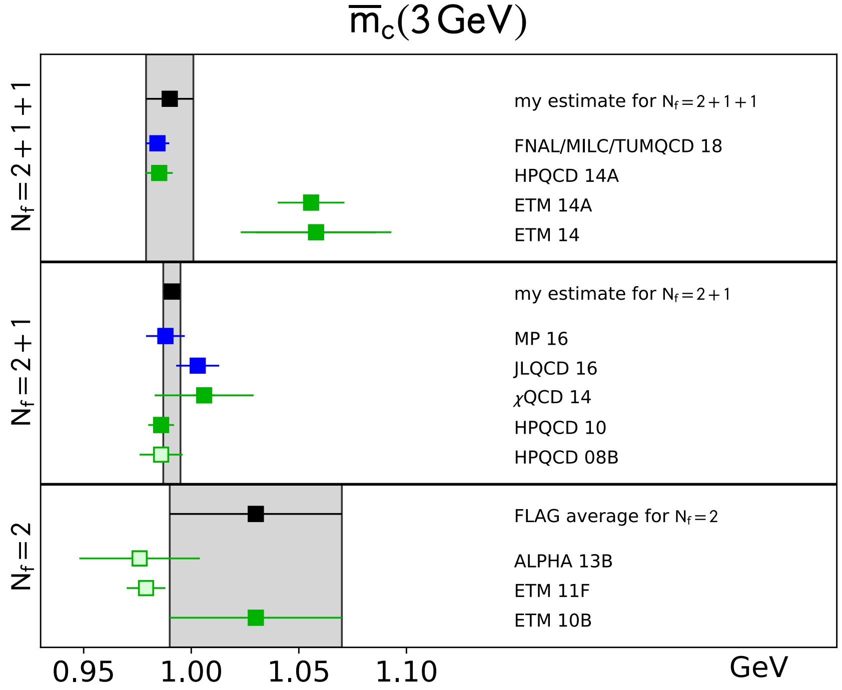
meson masses

$$\bar{m}_c(3\,GeV) = 1.003(10)\,GeV \qquad N_f = 2+1$$ [JLQCD 16]

$$= 0.988(9)\,GeV \qquad N_f = 2+1$$ [Maezawa&Petreczky 16]

moment method





- moment method (HPQCD 10, HPQCD 14, JLQCD 15B) (independent on mass RCs)

- based on charmed meson or baryon masses (and on mass RCs)

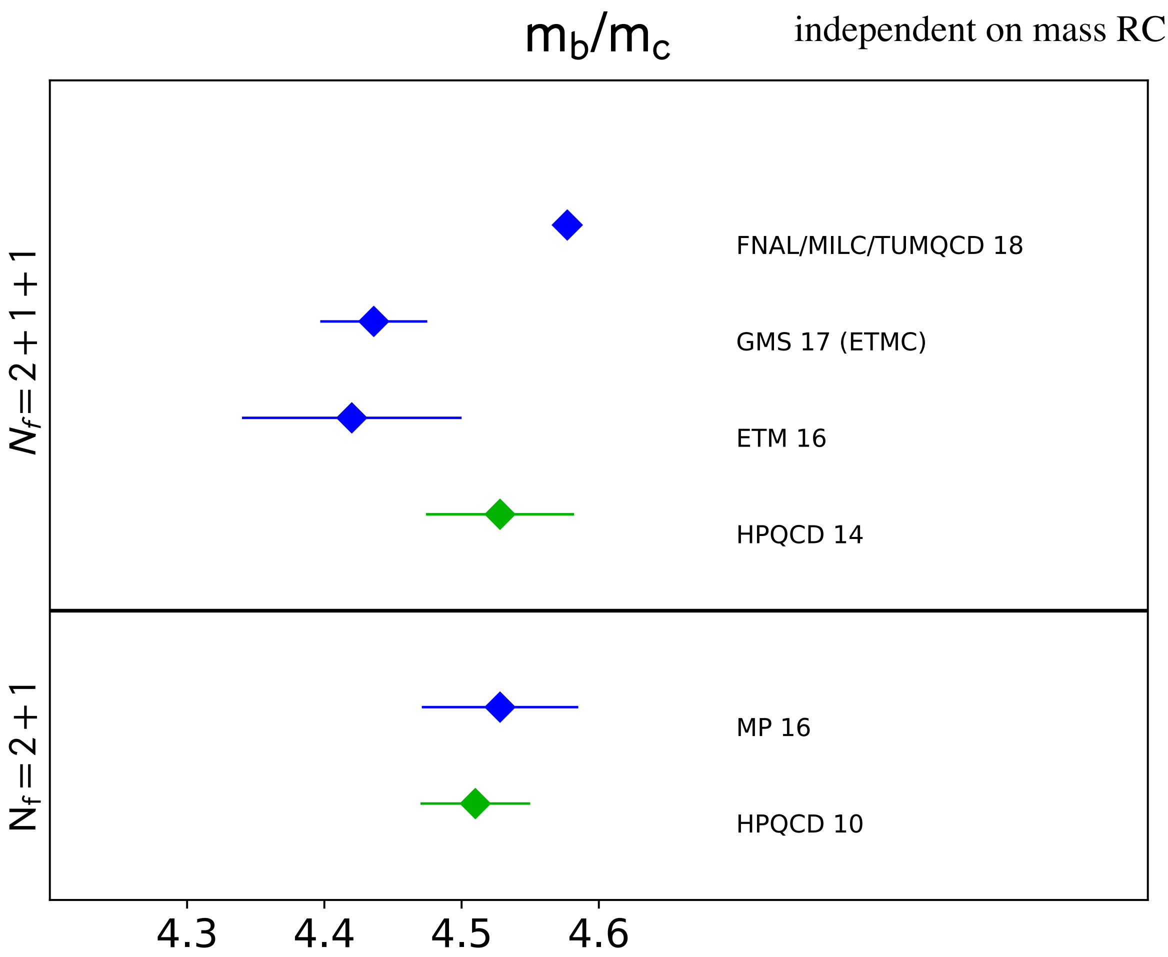
## my personal estimates

$$\bar{m}_c(3\,GeV) = 0.990(11)\,GeV \qquad N_f = 2+1+1 \qquad (\sim 1.1\%)$$

$$= 0.991(4)\,GeV \qquad N_f = 2+1 \qquad (\sim 0.4\%)$$

# ratio $m_b$ / $m_c$ from lattice QCD

independent on mass RCs



## FLAG-3 review (Nov. ’15)

$$\frac{\bar{m}_b}{\bar{m}_c} = 4.528(54) \qquad N_f = 2+1+1 \qquad [\text{HPQCD 14}]$$

$$\frac{\bar{m}_b}{\bar{m}_c} = 4.51(4) \qquad N_f = 2+1 \qquad [\text{HPQCD 10}]$$

## updates / new results

$$\frac{\bar{m}_b}{\bar{m}_c} = 4.577(9) \qquad N_f = 2+1+1 \qquad [\text{FNAL/MILC/TUMQCD 18}]$$

$$= 4.436(39) \qquad N_f = 2+1+1 \qquad [\text{GMS 17 (ETMC)}]$$

$$= 4.42(8) \qquad N_f = 2+1+1 \qquad [\text{ETMC 16}]$$

$$\frac{\bar{m}_b}{\bar{m}_c} = 4.528(57) \qquad N_f = 2+1 \qquad [\text{Maezawa\&Petreczky 16}]$$

## my personal estimates

$$\frac{\bar{m}_b}{\bar{m}_c} = 4.567(20) \qquad N_f = 2+1+1 \qquad (\sim 0.4\%)$$

$$= 4.516(33) \qquad N_f = 2+1 \qquad (\sim 0.7\%)$$

- **FLAG scrutiny**
- **full treatment of QED effects**

# HQE of PS and V heavy-light meson masses

$$M_{av} \equiv \frac{M_{PS} + 3M_V}{4} \quad \left(spin-averaged\right) \qquad \Delta M \equiv M_V - M_{PS} \quad \left(hyperfine\ splitting\right)$$

$$\frac{M_{av}}{m_h} = 1 + \frac{\overline{\Lambda}}{m_h} + \frac{\mu_\pi^2}{2m_h^2} + \frac{\rho_D^3 - \rho_{\pi\pi}^3 - \rho_S^3}{4m_h^3} + \frac{\sigma^4}{m_h^4} + O\left(\frac{1}{m_h^5}\right)$$

$$m_h \Delta M = \frac{2}{3} c_G \mu_G^2 + \frac{\rho_{\pi G}^3 + \rho_A^3 - \rho_{LS}^3}{3m_h} + \frac{\Delta\sigma^4}{m_h^2} + O\left(\frac{1}{m_h^3}\right)$$

$\mu_\pi^2, \mu_G^2, \rho_D^3, \rho_{LS}^3$ = dim-5 and dim-6 matrix elements appearing in $\Gamma_{sl}$ but in the static limit

$\overline{\Lambda}$ = (dim-4) binding energy of the light quark and gluons

$\rho_{\pi\pi}^3, \rho_S^3, \rho_{\pi G}^3, \rho_A^3$ = dim-6 (non-local) matrix elements not appearing in $\Gamma_{sl}$

$\sigma^4, \Delta\sigma^4$ = dim-7 matrix elements

* until 2004 few lattice determinations of HQE matrix elements only for $N_f = 0$ (quenched approx.)

[UKQCD ’96, Gimenez et al. ’97, Kronfeld&Simone ’00, Ali-Khan et al. ’00, JLQCD ’04]

* after 13 years the first unquenched calculation [Gambino, Melis, SS: PRD 96 (2017) 014511 [1704.06105]] followed by the second one [FNAL/MILC/TUMQCD: 1802.04248]

## which heavy-quark mass $m_h$ ?

the meaning of the HQE matrix elements and the size of the perturbative corrections in $\Gamma_{sl}$ or $M_{PS(V)}$ depend strongly on the definition of $m_h$

the natural choice is the **pole mass** (same meaning in QCD and HQET), but the perturbative series is not convergent (*n!* growth at large order n) and has an intrinsic IR renormalon ambiguity

## short-distance masses

- $\overline{MS}$ mass at a renormalization scale $\mu$: $\overline{\mathrm{m}}_h(\mu)$
- the mass in the 1S scheme [Hoang et al. '99] based on $\Upsilon(1S)$ and the potential subtracted mass [Beneke '98]

## explicit subtraction of leading (and subleading) IR renormalons

- the Renormalon Subtracted (RS) mass [Pineda '01]
- the MS-R mass [Hoang et al '08]
- the kinetic mass [Bigi et al. '97, Uraltsev '97] ⟶ Gambino, Melis, SS: PRD 96 (2017) 014511 [1704.06105]
- the Minimal RS mass [TUMQCD '18] ⟶ FNAL/MILC/TUMQCD: 1802.04248

# kinetic scheme

- it is based on the small velocity sum rules in which the separation between soft and hard effects, $\mu_{soft}$, is introduced as a cutoff over the excitation energy of the hadronic states [Bigi et al. ’97, Uraltsev ’97]

- the kinetic mass is defined by subtracting from the pole mass the perturbative contributions of the HQE parameters calculated from the SV sum rules cut at $\mu_{soft}$

- the kinetic mass enters the non-relativistic kinetic energy in the (renormalized) heavy-quark Hamiltonian

$$\tilde{m}_h \equiv m_h^{kin}\left(\mu_{soft}\right) = \mathrm{m}_h^{pole} - \left[\bar{\Lambda}\left(\mu_{soft}\right)\right]_{pert} - \frac{\left[\mu_\pi^2\left(\mu_{soft}\right)\right]_{pert}}{2\tilde{m}_h} - \frac{\left[\rho_D^3\left(\mu_{soft}\right)\right]_{pert}}{4\tilde{m}_h^2} - \dots$$

$$\left[\bar{\Lambda}\left(\mu_{soft}\right)\right]_{pert} = \frac{4}{3} C_F \frac{\alpha_s}{\pi} \mu_{soft} \left\{ 1 + \frac{\alpha_s}{\pi} \lambda_0 \right\}$$

[Czarnecki et al. ’98, Benson et al. ’03]

$$\left[\mu_\pi^2\left(\mu_{soft}\right)\right]_{pert} = \frac{3}{4} \mu_{soft} \left[\bar{\Lambda}\left(\mu_{soft}\right)\right]_{pert} - C_F \left(\frac{\alpha_s}{\pi}\right)^2 \beta_0 \mu_{soft}^2$$

$$\lambda_0 \equiv 2\beta_0 \left( \frac{8}{3} - \ln \frac{2\mu_{soft}}{m_h} \right) - C_A \left( \frac{\pi^2}{6} - \frac{13}{12} \right)$$

$$\left[\rho_D^3\left(\mu_{soft}\right)\right]_{pert} = \frac{1}{2} \mu_{soft}^2 \left[\bar{\Lambda}\left(\mu_{soft}\right)\right]_{pert} - \frac{8}{9} C_F \left(\frac{\alpha_s}{\pi}\right)^2 \beta_0 \mu_{soft}^3$$

$$\beta_0 = \frac{33 - 11 n_\ell}{12}$$

- leading and subleading IR renormalons subtracted → improvement of the perturbative convergence

# lattice QCD simulations

ETMC gauge configurations with $N_f = 2 + 1 + 1$ dynamical quarks (two light mass-degenerate quarks, strange and charm quarks close to their physical values)

TABLE I. Values of the valence-quark bare masses considered for the 15 ETMC gauge ensembles with $N_f = 2+1+1$ dynamical quarks (see Ref. [27]). $N_{\rm cfg}$ stands for the number of (uncorrelated) gauge configurations used in this work.

| ensemble | $\beta$ | $V/a^4$ | $N_{\rm cfg}$ | $a\mu_\ell$ | $a\mu_c$ | $a\mu_h > a\mu_c$ |
|---|---|---|---|---|---|---|
| $A30.32$ | 1.90 | $32^3 \times 64$ | 150 | 0.0030 | $\{0.21256, 0.25000, 0.29404\}$ | $\{0.34583, 0.40675, 0.47840, 0.56267, 0.66178, 0.77836, 0.91546\},$ |
| $A40.32$ | | | 150 | 0.0040 | | |
| $A50.32$ | | | 150 | 0.0050 | | |
| $A40.24$ | | $24^3 \times 48$ | 150 | 0.0040 | | |
| $A60.24$ | | | 150 | 0.0060 | | |
| $A80.24$ | | | 150 | 0.0080 | | |
| $A100.24$ | | | 150 | 0.0100 | | |
| $B25.32$ | 1.95 | $32^3 \times 64$ | 150 | 0.0025 | $\{0.18705, 0.22000, 0.25875\}$ | $\{0.30433, 0.35794, 0.42099, 0.49515, 0.58237, 0.68495, 0.80561\}$ |
| $B35.32$ | | | 150 | 0.0035 | | |
| $B55.32$ | | | 150 | 0.0055 | | |
| $B75.32$ | | | 75 | 0.0075 | | |
| $B85.24$ | | $24^3 \times 48$ | 150 | 0.0085 | | |
| $D15.48$ | 2.10 | $48^3 \times 96$ | 90 | 0.0015 | $\{0.14454, 0.0150, 0.19995\}$ | $\{0.23517, 0.27659, 0.32531, 0.38262, 0.45001, 0.52928, 0.62252\}$ |
| $D20.48$ | | | 90 | 0.0020 | | |
| $D30.48$ | | | 90 | 0.0030 | | |

$a \approx 0.06, 0.08, 0.09\ fm$ $\qquad$ $M_\pi \approx (210-450)\ MeV$ $\qquad$ $L \approx (2-3)\ fm$

ETMC: NPB '14

$m_\ell \approx (3-12)\, m_\ell^{phys}$ $\qquad$ $m_c \approx (0.7-1.1)\, m_c^{phys}$ $\qquad$ $m_h \approx (1.1-3.3)\, m_c^{phys} \approx (0.25-0.75)\, m_b^{phys}$

# extraction of ground-state PS and V meson masses

ground-state PS and V meson masses $M_{PS(V)}$ can be determined from the large-time plateaus of the effective mass

$$M^{eff}_{PS(V)}(t) = \mathrm{arcosh}\left[\frac{C_{PS(V)}(t-1)+C_{PS(V)}(t+1)}{2C_{PS(V)}(t)}\right] \xrightarrow[t\geq t^{PS(V)}_{\mathrm{min}}]{} M_{PS(V)}$$

Gaussian-smeared interpolating fields (and APE smearing of the gauge links) are adopted to suppress excited states

check with the Generalized EigenValue Problem method [Blossier et al. '09] using $C^{LL}$, $C^{SL}$, $C^{LS}$ and $C^{SS}$ correlators

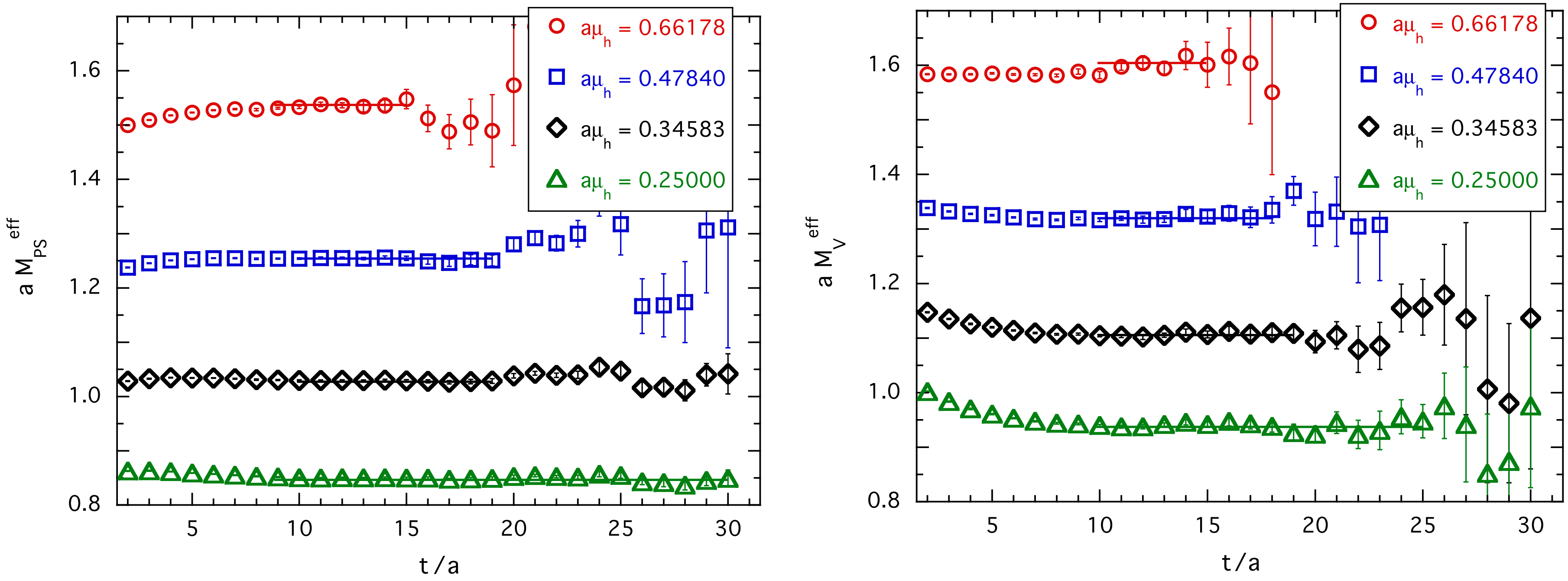


FIG. 2. Left panel: Effective masses of the correlator $C^{SL}_{PS}(t)$ calculated for various $(h\ell)$ mesons using Eq. (8) in the case of the ETMC gauge ensemble A40.32 (corresponding to a pion mass $\simeq$320 MeV). Right panel: The same as in the left panel, but for the vector correlator $C^{SL}_{V}(t)$. The solid lines identify the plateau region $t_{\mathrm{min}} \leq t \leq t_{\mathrm{max}}$ selected for each value of the heavy-quark mass.

# ETMC ratio method

the ETMC ratio method [Blossier et al. ’09] has been developed in order to reach the b-quark region from the charm one:

step 1: calculation of the observable around the charm scale employing dynamical simulations with controlled discretization effects

step 2: construction of appropriate ratios at increasing values of $m_h$ up to ~ 3 $m_c$; the ratios must go to 1 in the static limit

step 3: smooth interpolation of the ratios from $m_c$ to the static limit and evaluation of the observable at the b-quark mass

features/advantages

1. the same relativistic action setup is used for both light and heavy quarks
2. extra simulations in the static limit are not necessary
3. discretization effects are reduced in the ratios (better control of their continuum limit)

applied both at $N_f = 2$ and $N_f = 2 + 1 + 1$ for evaluating:

- the b-quark mass [JHEP ’09, JHEP ’12, PRD ’16]
- the leptonic decay constants of B and $B_s$ mesons [JHEP ’09, JHEP ’12, PRD ’16] as well as of B* and $B^*_s$ mesons [PRD ’17]
- the bag parameters for neutral B-meson oscillations [JHEP ’14]

* interpolation of the lattice data at a sequence of heavy-quark masses $\tilde{m}_h^{(n+1)} = \lambda \tilde{m}_h^{(n)} \quad (n = 1,2,\ldots K)$

* $\lambda$ is tuned so that $\tilde{m}_h^{(K+1)} = \tilde{m}_b^{phys}$ starting from $\tilde{m}_h^{(1)} = \tilde{m}_c^{phys}$

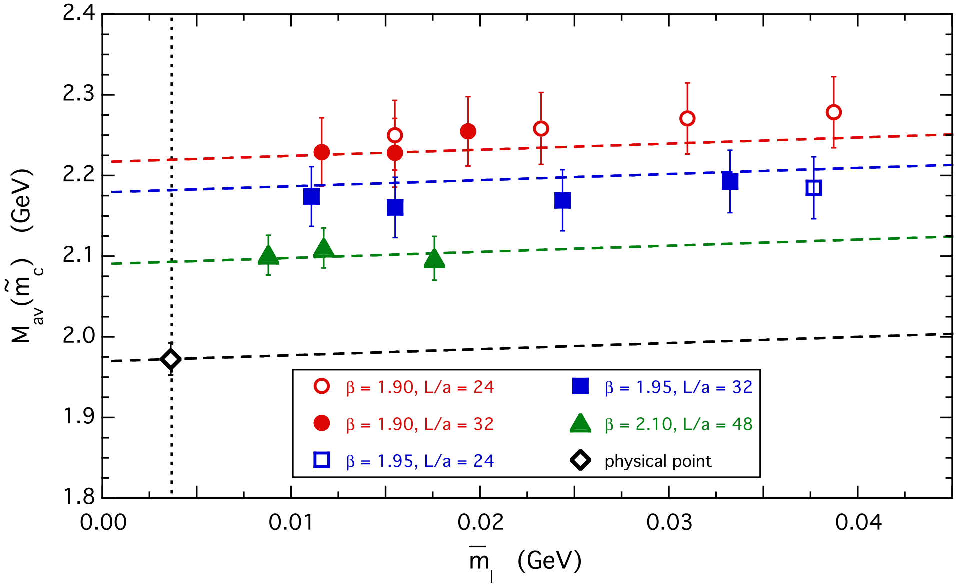
# spin-averaged masses

**- triggering point at the charm scale:**

extrapolation to the physical pion point and to the continuum limit

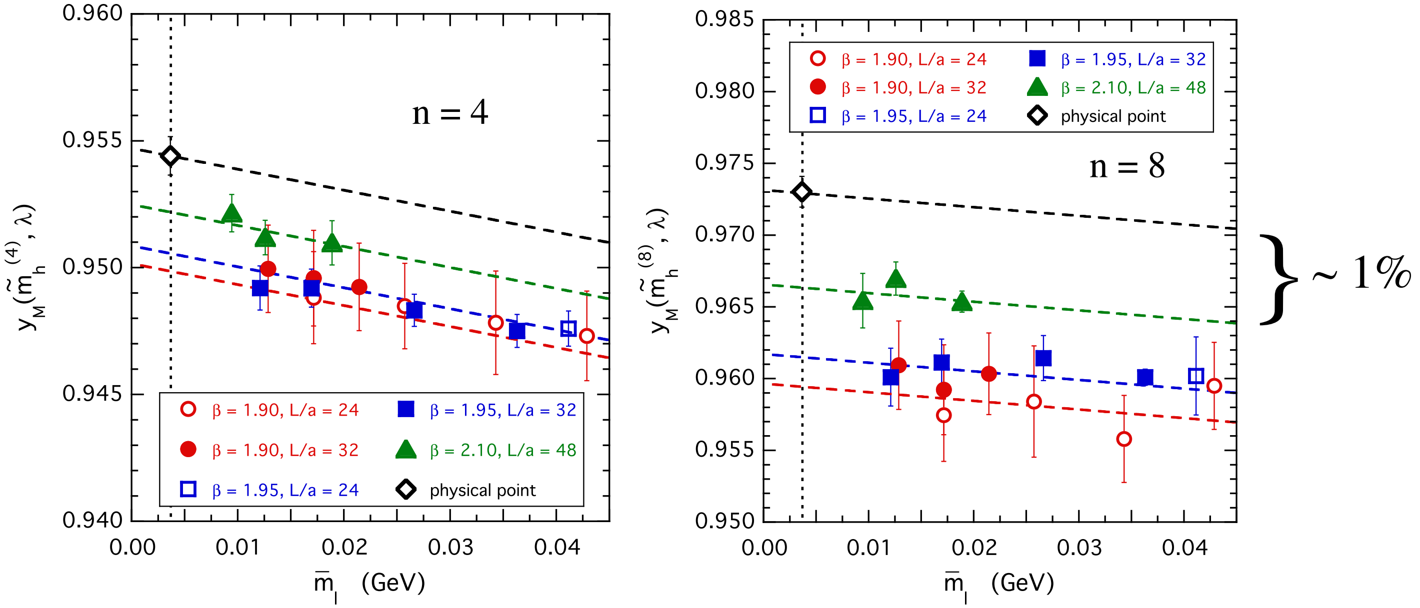
$$M_{av}\left[\tilde{m}_c^{phys}=1.219(41)\,GeV\right]=1.967(25)\,GeV$$

$$\text{PDG:}\quad \frac{M_D+3M_{D^*}}{4}=1.973\,GeV$$



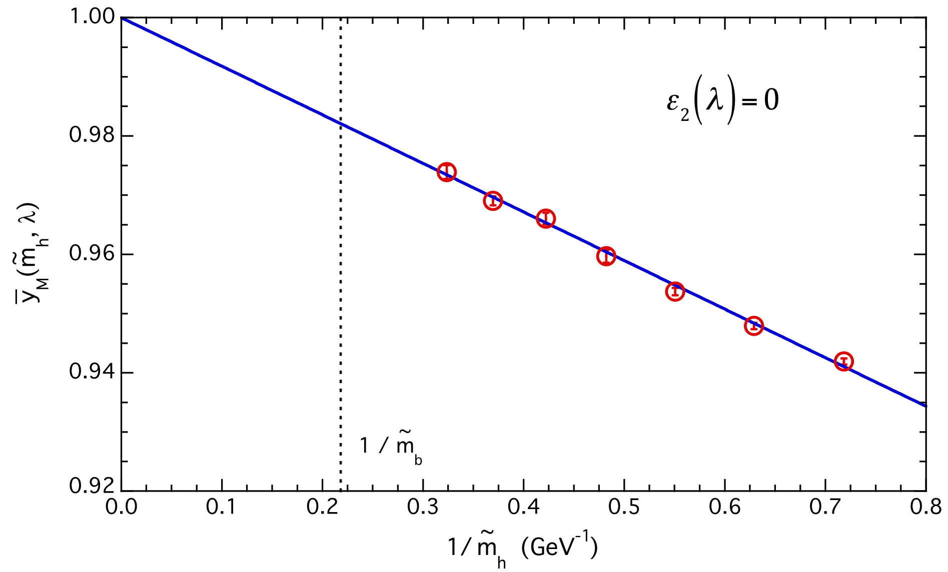
**- ratios of spin-averaged meson masses:**

$$y_M\left(\tilde{m}_h^{(n)},\lambda\right)\equiv\frac{M_{av}\left(\tilde{m}_h^{(n)}\right)}{M_{av}\left(\tilde{m}_h^{(n-1)}\right)}\frac{\tilde{m}_h^{(n-1)}}{\tilde{m}_h^{(n)}}=\frac{1}{\lambda}\frac{M_{av}\left(\tilde{m}_h^{(n)}\right)}{M_{av}\left(\tilde{m}_h^{(n-1)}\right)}\qquad (n=2,3,\ldots)\qquad y_M=1+O\left(\frac{1}{\tilde{m}_h}\right)$$



discretization effects under very good control even at $m_h \sim 3\, m_c$

## - extrapolation imposing the known static limit:



$$y_M(\tilde{m}_h,\lambda)=1+\frac{\varepsilon_1(\lambda)}{\tilde{m}_h}+\frac{\varepsilon_2(\lambda)}{\tilde{m}_h^2}+O\left(\frac{1}{\tilde{m}_h^3}\right)$$

- correlations between lattice points taken into account through the covariance matrix

- a bootstrap analysis is used for taking care of the various determinations of all the input parameters of the analysis (lattice spacing, physical quark masses, RCs, ...)

**- chain equation:** $$M_{av}\left(\tilde{m}_h^{(1)}=\tilde{m}_c^{phys}\right)\cdot y_M\left(\tilde{m}_h^{(2)},\lambda\right)\cdot y_M\left(\tilde{m}_h^{(3)},\lambda\right)\ldots\cdot y_M\left(\tilde{m}_h^{(K+1)},\lambda\right)=\lambda^K M_{av}\left(\tilde{m}_h^{(K+1)}\right)$$

$\tilde{m}_b^{phys}=\lambda^K\tilde{m}_c^{phys}$ obtained (iteratively) by imposing that $M_{av}\left(\tilde{m}_h^{(K+1)}\right)=\frac{M_B+3M_{B^*}}{4}$ after K=10 steps

<u>ETMC '14 value</u>: $\tilde{m}_c^{phys}=1.219(41)\,GeV$ → $\bar{\mathrm{m}}_c(\bar{\mathrm{m}}_c)=1.348(46)\,GeV$ error ~ 3 %

$r^{kin}=\frac{\tilde{m}_b^{phys}}{\tilde{m}_c^{phys}}=3.780(34)$ → $\bar{\mathrm{r}}=\frac{\bar{m}_b}{\bar{m}_c}=4.436(39)$ correlation $\rho\left(\tilde{m}_c^{phys},\tilde{m}_b^{phys}\right)\cong 1$

$\tilde{m}_b^{phys}=4.605(120)_{stat}(57)_{syst}\,GeV=4.605(132)\,GeV$ → $\bar{\mathrm{m}}_b(\bar{\mathrm{m}}_b)=4.257(120)\,GeV$

* consistent with other ETMC determinations of the b-quark mass and with the FLAG-3 average within 1σ

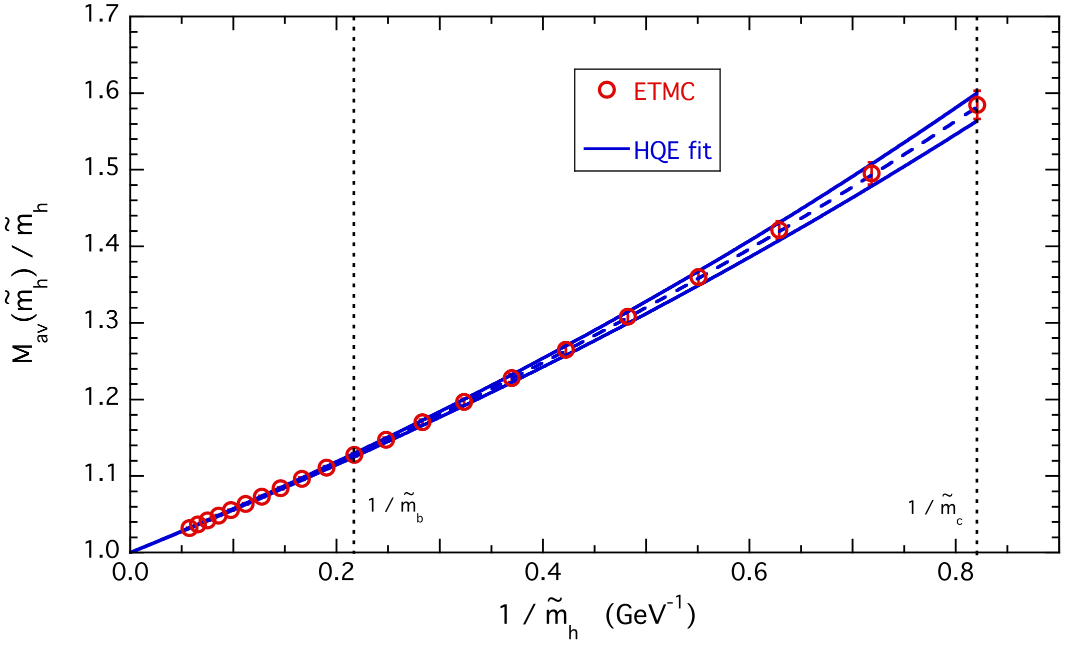
* fits of semileptonic moments: $\tilde{m}_b^{phys}=4.546(21)\,GeV$ [Gambino et al. '16]

- **extension of the chain equation:**

$$\frac{M_{av}\left(\tilde{m}_h^{(n)}\right)}{\tilde{m}_h^{(n)}} = \frac{\prod_{i=2}^{n} y_M\left(\tilde{m}_h^{(i)},\lambda\right)}{\prod_{i=2}^{\infty} y_M\left(\tilde{m}_h^{(i)},\lambda\right)}$$

up to n~20 $\rightarrow$ $\tilde{m}_h \sim 4\,\tilde{m}_b$

very precise (and correlated) data



**dimension-6 fit**

$$\frac{M_{av}\left(\tilde{m}_h\right)}{\tilde{m}_h} = 1 + \frac{\overline{\Lambda}}{\tilde{m}_h} + \frac{\mu_\pi^2}{2\tilde{m}_h^2} + \frac{\rho_D^3 - \rho_{\pi\pi}^3 - \rho_S^3}{4\tilde{m}_h^3}$$

$$\overline{\Lambda} = 0.551\left(13\right)_{stat}\left(2\right)_{syst}\ GeV$$

$$\mu_\pi^2 = 0.314\left(14\right)_{stat}\left(2\right)_{syst}\ GeV^2$$

$$\rho_D^3 - \rho_{\pi\pi}^3 - \rho_S^3 = 0.174\left(12\right)_{stat}\left(2\right)_{syst}\ GeV^3$$

**dimension-7 fit**

$$\frac{M_{av}\left(\tilde{m}_h\right)}{\tilde{m}_h} = 1 + \frac{\overline{\Lambda}}{\tilde{m}_h} + \frac{\mu_\pi^2}{2\tilde{m}_h^2} + \frac{\rho_D^3 - \rho_{\pi\pi}^3 - \rho_S^3}{4\tilde{m}_h^3} + \frac{\sigma^4}{\tilde{m}_h^4}$$

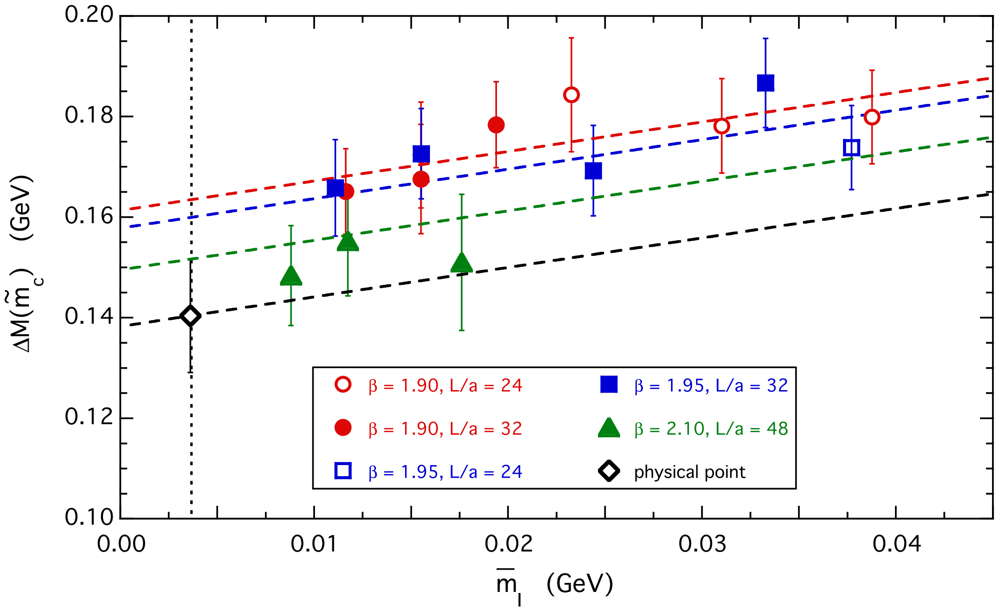
$$\overline{\Lambda} = 0.552\left(13\right)_{stat}\left(2\right)_{syst}\ GeV$$

$$\mu_\pi^2 = 0.325\left(17\right)_{stat}\left(2\right)_{syst}\ GeV^2$$

$$\rho_D^3 - \rho_{\pi\pi}^3 - \rho_S^3 = 0.133\left(34\right)_{stat}\left(6\right)_{syst}\ GeV^3$$

$$\sigma^4 = 0.0072\left(55\right)_{stat}\left(10\right)_{syst}\ GeV^4$$

# hyperfine splitting



**- triggering point at the charm scale:**

extrapolation to the physical pion point and to the continuum limit

$$\Delta M\left[\tilde m_c^{phys} = 1.219\left(41\right)GeV\right] = 140\left(11\right)MeV$$

PDG: $M_{D^*} - M_D = 141.4\,MeV$

**- ratios of hyperfine splittings:**

$$\Delta y_M\left(\tilde m_h^{(n)},\lambda\right) \equiv \frac{\tilde m_h^{(n)}\Delta M\left(\tilde m_h^{(n)}\right)}{\tilde m_h^{(n-1)}\Delta M\left(\tilde m_h^{(n-1)}\right)}\,\frac{c_G\left(\tilde m_h^{(n-1)},\tilde m_b\right)}{c_G\left(\tilde m_h^{(n)},\tilde m_b\right)} \qquad \left(n=2,3,\ldots\right) \qquad \Delta y_M = 1 + O\left(\frac{1}{\tilde m_h}\right)$$

$$\Big\downarrow$$

$$\tilde m_h \Delta M\left(\tilde m_h\right) = \frac{2}{3} c_G\left(\tilde m_h,\tilde m_b\right)\mu_G^2\left(\tilde m_b\right) + O\left(\frac{1}{\tilde m_h}\right)$$

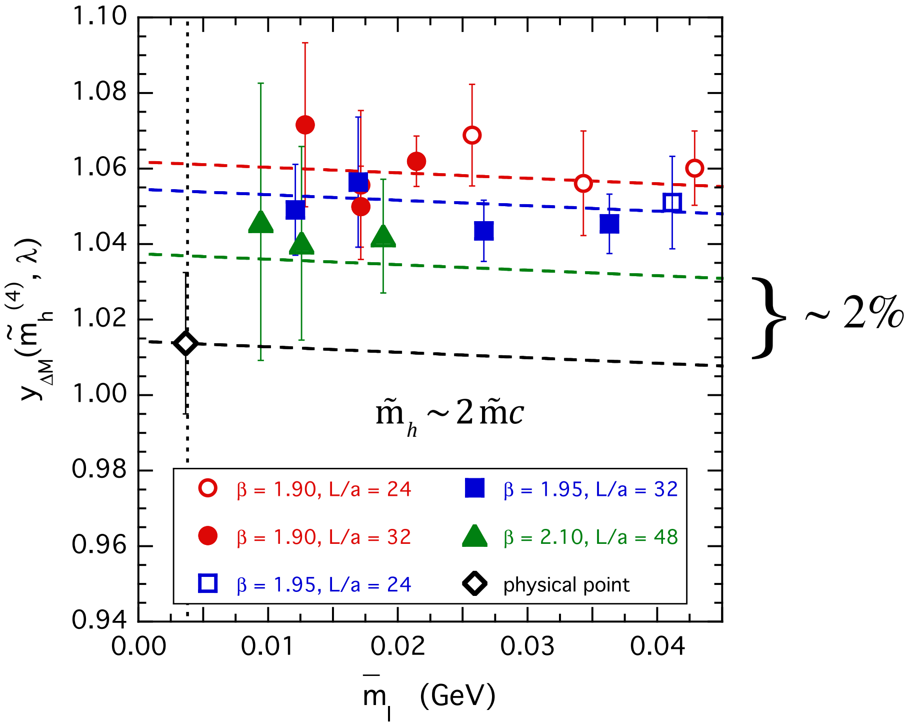
**- $c_G$ is the short-distance Wilson coefficient of the HQE chromomagnetic operator:**

$$c_G = \bar c_G\left(\bar m_h\right)\cdot\mathbb{R}\left(\bar m_h,\bar m_b\right)\cdot\frac{\tilde m_h\left(\bar m_h\right)}{m_h^{pole}\left(\bar m_h\right)} \quad \rightarrow \quad \text{known up to } \alpha_s^2$$

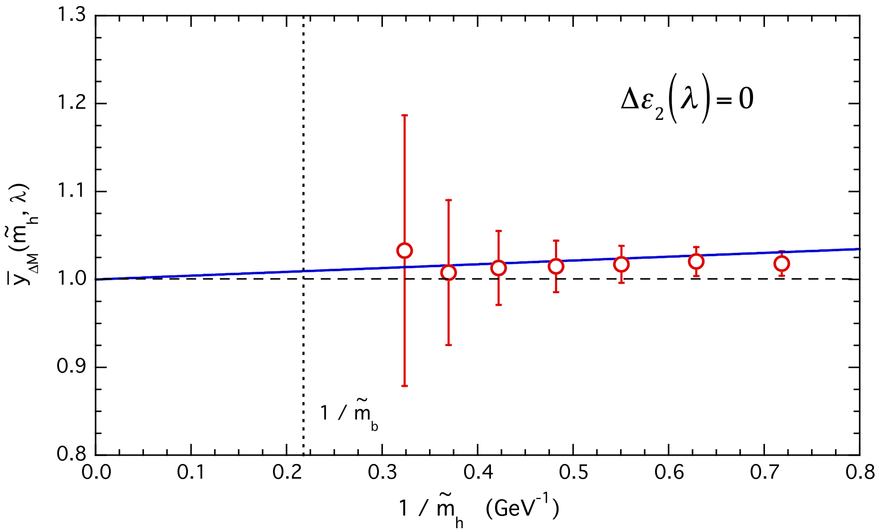
matches HQE with QCD

evolution in the $\overline{\text{MS}}$ scheme

replace the pole mass with the kinetic one in the HQE discretization effects on ratios of hyperfine splittings larger than the corresponding ones of the spin-averaged masses, but still under good control



**- extrapolation imposing the known static limit:**



$$y_{\Delta M}\left(\tilde{m}_h,\lambda\right)=1+\frac{\Delta\varepsilon_1\left(\lambda\right)}{\tilde{m}_h}+\frac{\Delta\varepsilon_2\left(\lambda\right)}{\tilde{m}_h^2}+O\left(\frac{1}{\tilde{m}_h^3}\right)$$

correlations between lattice points
and bootstrap samplings
taken into account

**- using the chain equation, at the b-quark mass:**

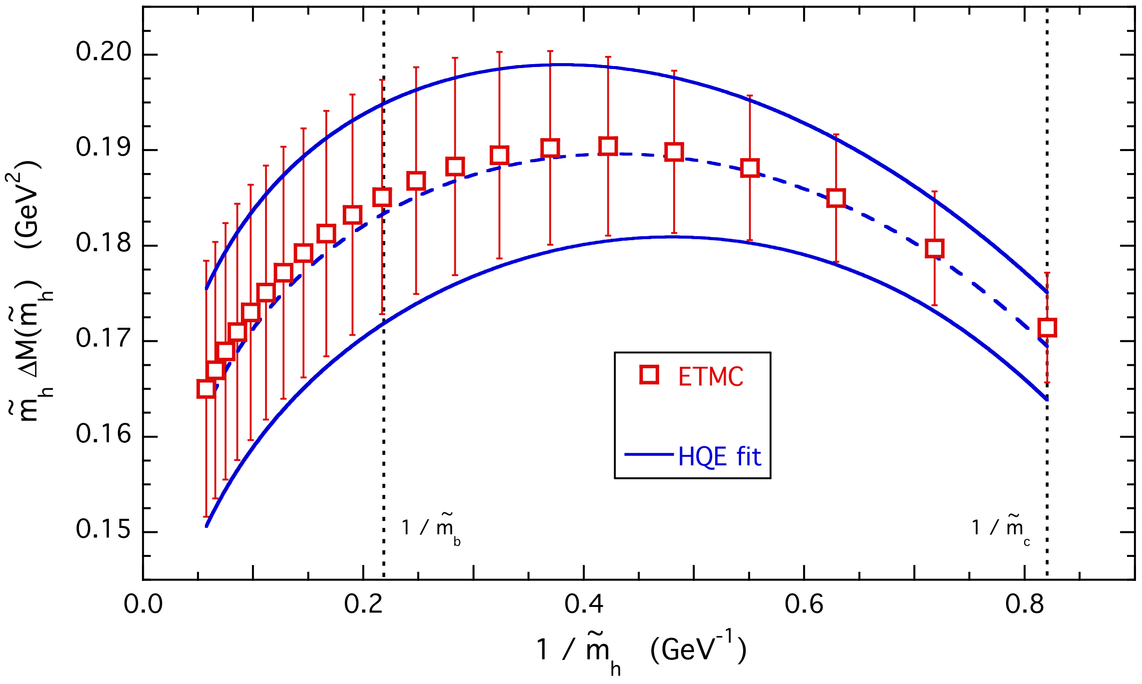
$$\Delta M\left(\tilde{m}_b^{phys}\right)=40.2\left(2.1\right)MeV$$

PDG: $M_{B^*}-M_B=45.42\left(26\right)MeV$

**- extension of the chain equation:**

$$\frac{\tilde{m}_h^{(n)} \Delta M\left(\tilde{m}_h^{(n)}\right)}{c_G\left(\tilde{m}_h^{(n)},\tilde{m}_b\right)} = \frac{\tilde{m}_c \Delta M\left(\tilde{m}_c\right)}{c_G\left(\tilde{m}_c,\tilde{m}_b\right)} \cdot \prod_{i=2}^{n} y_{\Delta M}\left(\tilde{m}_h^{(i)},\lambda\right)$$

up to n~20 → $\tilde{m}_h \sim 4\,\tilde{m}_b$



**dimension-6 fit**

$$\tilde{m}_h \Delta M\left(\tilde{m}_h\right) = \frac{2}{3} c_G\left(\tilde{m}_h,\tilde{m}_b\right) \mu_G^2\left(\tilde{m}_b\right) + \frac{\rho_{\pi G}^3 + \rho_A^3 - \rho_{LS}^3}{3\tilde{m}_h}$$

$$\mu_G^2\left(\tilde{m}_b\right) = 0.250(18)_{stat}\,(8)_{syst}\ GeV^2$$
$$\rho_{\pi G}^3 + \rho_A^3 - \rho_{LS}^3 = -0.143(57)_{stat}\,(21)_{syst}\ GeV^3$$

**dimension-7 fit**

$$\tilde{m}_h \Delta M\left(\tilde{m}_h\right) = \frac{2}{3} c_G\left(\tilde{m}_h,\tilde{m}_b\right) \mu_G^2\left(\tilde{m}_b\right) + \frac{\rho_{\pi G}^3 + \rho_A^3 - \rho_{LS}^3}{3\tilde{m}_h} + \frac{\Delta\sigma^4}{\tilde{m}_h^2}$$

$$\mu_G^2\left(\tilde{m}_b\right) = 0.254(20)_{stat}\,(9)_{syst}\ GeV^2$$
$$\rho_{\pi G}^3 + \rho_A^3 - \rho_{LS}^3 = -0.173(74)_{stat}\,(25)_{syst}\ GeV^3$$
$$\Delta\sigma^4 = 0.0092(58)_{stat}\,(14)_{syst}\ GeV^4$$

# two brief comments

- **in the (so-called) BPS limit** [Uraltsev '04] **one has:** $\mu_\pi^2 = \mu_G^2\left(\tilde{m}_b\right)$ and $\rho_D^3 + \rho_{LS}^3 = 0$ $\left[\vec{\sigma}\cdot\vec{\pi}\middle|B\right\rangle = 0\Big]$

$$\mu_\pi^2 - \mu_G^2\left(\tilde{m}_b\right) = 0.064\left(19\right)GeV^2 \qquad \left[\sim 3\sigma\right]$$

$$\rho_D^3 + \rho_{LS}^3 \geq 0.317\left(65\right)GeV^3 \qquad \left[\sim 5\sigma\right]$$

- **difference between HQE matrix elements in the B-meson and in the static limit**

$$\mu_\pi^2\Big|_B = \mu_\pi^2 - \frac{\rho_{\pi\pi}^3 + \frac{1}{2}\rho_{\pi G}^3}{\tilde{m}_b} + O\left(\frac{1}{\tilde{m}_b^2}\right) \qquad \rho_{\pi\pi}^3 + \frac{1}{2}\rho_{\pi G}^3 = 0 \quad \left(\text{in the BPS limit}\right)$$

$$\mu_G^2\left(\tilde{m}_b\right)\Big|_B = \mu_G^2\left(\tilde{m}_b\right) + \frac{\rho_S^3 + \rho_A^3 + \frac{1}{2}\rho_{\pi G}^3}{\tilde{m}_b} + O\left(\frac{1}{\tilde{m}_b^2}\right)$$

using our $\mu_\pi^2 = 0.325\left(17\right)GeV^2$ and $\mu_\pi^2\Big|_B = 0.432\left(68\right)GeV^2$ from fits of semileptonic moments: $\rho_{\pi\pi}^3 + \frac{1}{2}\rho_{\pi G}^3 = -0.51\left(35\right)GeV^3$

$$\rho_S^3 + \rho_A^3 + \rho_{\pi G}^3 + \rho_{\pi\pi}^3 \geq 0 \quad \longrightarrow \quad \rho_S^3 + \rho_A^3 + \frac{1}{2}\rho_{\pi G}^3 \geq 0.51\left(35\right)GeV^3$$

= only in the BPS limit

$$\mu_G^2\left(\tilde{m}_b\right)\Big|_B \geq \mu_G^2\left(\tilde{m}_b\right) + 0.11\left(8\right)GeV^2 = 0.25\left(2\right)GeV^2 + 0.11\left(8\right)GeV^2 = 0.36\left(8\right)GeV^2$$

$$\mu_G^2\left(\tilde{m}_b\right)\Big|_B = 0.35\left(7\right)GeV^2 \quad \left(\text{from } B^* - B \text{ splitting}\right)$$

~ 30 % difference between the static limit
and the B-meson CMO matrix element

## heavy-light (u/d) mesons

### dimension-6 fit

$$\bar{\Lambda} = 0.551(13)_{stat}\,(2)_{syst}\ GeV$$
$$\mu_\pi^2 = 0.314(14)_{stat}\,(2)_{syst}\ GeV^2$$
$$\rho_D^3 - \rho_{\pi\pi}^3 - \rho_S^3 = 0.174(12)_{stat}\,(2)_{syst}\ GeV^3$$

$$\mu_G^2(\tilde{m}_b) = 0.250(18)_{stat}\,(8)_{syst}\ GeV^2$$
$$\rho_{\pi G}^3 + \rho_A^3 - \rho_{LS}^3 = -0.143(57)_{stat}\,(21)_{syst}\ GeV^3$$

### dimension-7 fit

$$\bar{\Lambda} = 0.552(13)_{stat}\,(2)_{syst}\ GeV$$
$$\mu_\pi^2 = 0.325(17)_{stat}\,(2)_{syst}\ GeV^2$$
$$\rho_D^3 - \rho_{\pi\pi}^3 - \rho_S^3 = 0.133(34)_{stat}\,(6)_{syst}\ GeV^3$$
$$\sigma^4 = 0.0072(55)_{stat}\,(10)_{syst}\ GeV^4$$

$$\mu_G^2(\tilde{m}_b) = 0.254(20)_{stat}\,(9)_{syst}\ GeV^2$$
$$\rho_{\pi G}^3 + \rho_A^3 - \rho_{LS}^3 = -0.173(74)_{stat}\,(25)_{syst}\ GeV^3$$
$$\Delta\sigma^4 = 0.0092(58)_{stat}\,(14)_{syst}\ GeV^4$$

## heavy-strange mesons

### dimension-6 fit

$$\bar{\Lambda} = 0.637(15)_{stat}\,(6)_{syst}\ GeV$$
$$\mu_\pi^2 = 0.414(19)_{stat}\,(8)_{syst}\ GeV^2$$
$$\rho_D^3 - \rho_{\pi\pi}^3 - \rho_S^3 = 0.281(20)_{stat}\,(9)_{syst}\ GeV^3$$

$$\mu_G^2(\tilde{m}_b) = 0.299(14)_{stat}\,(5)_{syst}\ GeV^2$$
$$\rho_{\pi G}^3 + \rho_A^3 - \rho_{LS}^3 = -0.289(46)_{stat}\,(14)_{syst}\ GeV^3$$

### dimension-7 fit

$$\bar{\Lambda} = 0.636(15)_{stat}\,(6)_{syst}\ GeV$$
$$\mu_\pi^2 = 0.431(21)_{stat}\,(10)_{syst}\ GeV^2$$
$$\rho_D^3 - \rho_{\pi\pi}^3 - \rho_S^3 = 0.204(20)_{stat}\,(18)_{syst}\ GeV^3$$
$$\sigma^4 = 0.0128(30)_{stat}\,(30)_{syst}\ GeV^4$$

$$\mu_G^2(\tilde{m}_b) = 0.303(15)_{stat}\,(5)_{syst}\ GeV^2$$
$$\rho_{\pi G}^3 + \rho_A^3 - \rho_{LS}^3 = -0.359(58)_{stat}\,(14)_{syst}\ GeV^3$$
$$\Delta\sigma^4 = 0.0272(55)_{stat}\,(11)_{syst}\ GeV^4$$

## SU(3) breaking

$\sim 15\%$ for $\bar{\Lambda}$

$\sim 25\%$ for $\mu_\pi^2$

$\sim 40\%$ for $\rho_D^3 - \rho_{\pi\pi}^3 - \rho_S^3$

$\sim 15\%$ for $\mu_G^2$

$\sim 50\%$ for $\rho_{\pi G}^3 + \rho_A^3 - \rho_{LS}^3$

$$M_{av}\left[\tilde{m}_b^{phys}\right] = 5.403(12)\,GeV$$
$$\frac{M_{B_s} + 3M_{B_s^*}}{4} = 5.403(2)\,GeV \quad (\mathrm{PDG})$$

$$\Delta M\left[\tilde{m}_b^{phys}\right] = 45.1(1.1)\,MeV$$
$$M_{B_s^*} - M_{B_s} = 46.1(1.5)\,MeV \quad (\mathrm{PDG})$$

# open issue

- conversion to the kinetic scheme at the charm mass

$$a\hat{m}_c \xrightarrow[RI-MOM]{} \frac{Z_m(1/a)}{a}\left(a\hat{m}_c\right) \xrightarrow[\overline{MS}]{} \overline{m}_c(\mu) \xrightarrow[\text{kinetic scheme}]{} \tilde{m}_c\left(\mu_{soft}\right)$$

$\mu_{soft} = 1\ GeV$

bare charm mass
in lattice units

non-perturbative

$\alpha_s^3$

$\alpha_s^2$

- the kinetic mass should depend only on the Wilsonian cut $\mu_{soft}$
- instead, because of perturbative truncations it has a slight dependence on $\mu$

estimate: $\left(\delta\tilde{m}_c\right)_{conv} \approx 40\ MeV$

$$\tilde{m}_b = 4.605(132)(150)_{conv}\ GeV$$

$$\overline{\Lambda} = 0.552(13)(22)_{conv}\ GeV$$

$$\mu_\pi^2 = 0.321(17)(27)_{conv}\ GeV^2$$

$$\mu_G^2\left(\tilde{m}_b\right) = 0.253(21)(13)_{conv}\ GeV^2$$

$$\rho_D^3 - \rho_{\pi\pi}^3 - \rho_S^3 = 0.153(30)(17)_{conv}\ GeV^3$$

$$\rho_{\pi G}^3 + \rho_A^3 - \rho_{LS}^3 = -0.158(71)(45)_{conv}\ GeV^3$$

averages over dimension-6
and dimension-7 fits

ratio of bottom/charm quark masses
slightly sensitive to $()_{conv}$

$$r^{kin} = \frac{\tilde{m}_b}{\tilde{m}_c} = 3.78(4)(2)_{conv}$$

correlation $\rho\left(\tilde{m}_c, \tilde{m}_b\right) \cong 1$

***** removal of the $(\ )_{conv}$ uncertainty → conversion to the kinetic scheme at $\alpha_s^3$ *****

# Minimal Renormalon Subtraction scheme

[TUMQCD Coll.: Brambilla, Komijani, Kronfeld and Vairo: PRD '18]

* it is based on the subtraction of the leading IR renormalon in a way which is independent of the renormalization point μ

$$m_{pole} = \bar{m}\left(1+\sum_{n=0}^{N} r_n \alpha_s^{n+1}(\bar{m}) + O\left(\alpha_s^{N+2}\right)\right)$$

$r_n$ grows like n! at large n

$$m_{pole} = \bar{m}\left(1+\sum_{n=0}^{\infty}\left[r_n - R_n\right]\alpha_g^{n+1}(\bar{m})\right) + \delta m_{IR} + I_{MRS}(\bar{m})$$

$R_n$ compensates the large-n behavior of $r_n$

$$I_{MRS}(\mu) = \frac{R_0}{2\beta_0}\mu\int_0^1 dz\,\frac{e^{-z/2\beta_0\alpha_g(\mu)}}{(1-z)^{1+b}}$$

finite piece

$\alpha_g =$ strong coupling in the geometric scheme

$b = \beta_1/2\beta_0^2$

$$\delta m_{IR} = -(-)^b\frac{R_0}{2^{1+b}\beta_0}\Gamma(-b)\Lambda_{\overline{MS}}$$

it contains the leading IR renormalon and
it is independent of μ

MRS mass: $\boxed{m_{MRS} = m_{pole} - \delta m_{IR}}$

* the MRS mass keeps the advantages of the pole mass (IR finite, gauge invariant at each order, independent on μ), while avoiding its ambiguity

* the subtraction $\delta m_{IR}$ is a small correction with respect to $m_{MRS}$, i.e. $\delta m_{IR} << m_{MRS}$ preserves the matching of HQET with QCD

* its relation with $\overline{\text{MS}}$-mass is known up to O($\alpha_s^4$) and the convergence is stable $m_{\rm MRS}/\overline{m} = (1.133, 1.131, 1.132, 1.132)$

# FNAL/MILC/TUMQCD gauge ensembles

TABLE II. Valence-quark masses used in each ensemble. The first two columns identify the ensemble. The third column gives the lightest valence-quark mass in units of the sea strange-quark mass. (The full set of light valence-quark masses is listed in the text.) The fourth column shows the heavy valence-quark masses in units of the sea charm-quark mass. The last column shows the number of configurations and the number of source time slices used on each.

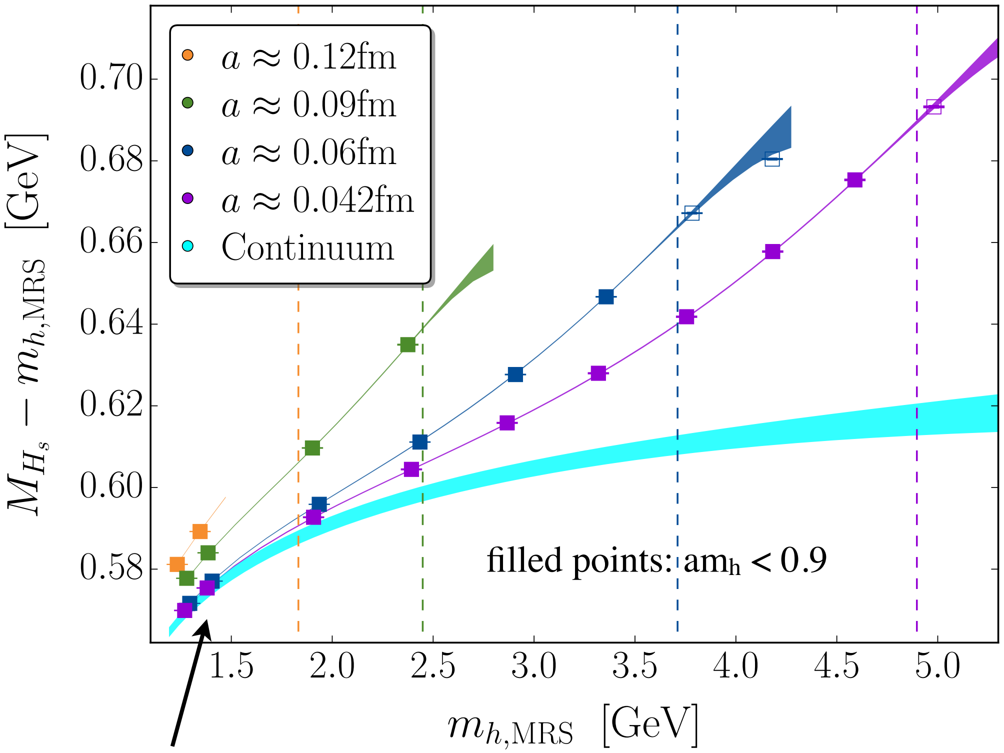
| ≈ $a$ (fm) | Key | $m_{\rm min}/m'_s$ | $m_h/m'_c$ | $N_{\rm conf} \times N_{\rm src}$ |
|---|---|---|---|---|
| 0.15 | $m_s/5$ | 0.1 | $\{0.9, 1.0\}$ | $1020 \times 4$ |
| 0.15 | $m_s/10$ | 0.1 | $\{0.9, 1.0\}$ | $1000 \times 4$ |
| 0.15 | physical | 0.037 | $\{0.9, 1.0\}$ | $1000 \times 4$ |
| 0.12 | $m_s/5$ | 0.1 | $\{0.9, 1.0\}$ | $1040 \times 4$ |
| 0.12 | unphysA | 0.1 | $\{0.9, 1.0\}$ | $1020 \times 4$ |
| 0.12 | small | 0.1 | $\{0.9, 1.0\}$ | $1020 \times 4$ |
| 0.12 | $m_s/10$ | 0.1 | $\{0.9, 1.0\}$ | $1000 \times 4$ |
| 0.12 | large | 0.1 | $\{0.9, 1.0\}$ | $1028 \times 4$ |
| 0.12 | unphysB | 0.1 | $\{0.9, 1.0\}$ | $1020 \times 4$ |
| 0.12 | unphysC | 0.1 | $\{0.9, 1.0\}$ | $1020 \times 4$ |
| 0.12 | unphysD | 0.1 | $\{0.9, 1.0\}$ | $1020 \times 4$ |
| 0.12 | unphysE | 0.1 | $\{0.9, 1.0\}$ | $1020 \times 4$ |
| 0.12 | unphysF | 0.1 | $\{0.9, 1.0\}$ | $1020 \times 4$ |
| 0.12 | unphysG | 0.1 | $\{0.9, 1.0\}$ | $1020 \times 4$ |
| 0.12 | physical | 0.037 | $\{0.9, 1.0\}$ | $999 \times 4$ |
| 0.09 | $m_s/5$ | 0.1 | $\{0.9, 1.0\}$ | $1005 \times 4$ |
| 0.09 | $m_s/10$ | 0.1 | $\{0.9, 1.0\}$ | $999 \times 4$ |
| 0.09 | physical | 0.033 | $\{0.9, 1.0, 1.5, 2.0, 2.5, 3.0\}$ | $484 \times 4$ |
| 0.06 | $m_s/5$ | 0.05 | $\{0.9, 1.0\}$ | $1016 \times 4$ |
| 0.06 | $m_s/10$ | 0.05 | $\{0.9, 1.0, 2.0, 3.0, 4.0\}$ | $572 \times 4$ |
| 0.06 | physical | 0.036 | $\{0.9, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5\}$ | $842 \times 6$ |
| 0.042 | $m_s/5$ | 0.036 | $\{0.9, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5\}$ | $1167 \times 6$ |
| 0.042 | physical | 0.037 | $\{0.9, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0\}$ | $420 \times 6$ |
| 0.03 | $m_s/5$ | 0.2 | $\{0.9, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0\}$ | $724 \times 4$ |

[arXiv: 1712.09262 and 1802.04248]

- HISQ action for fermions
- tadpole-improved one-loop Symanzik gauge action
- total no. of confs ~ 20 000
- 5 ensembles at the physical pion point
- 6 ensembles used with valence masses above the (physical) charm one

$$M_{PS}\left(m_h,m_x;a\right) = m_{h,MRS} + \bar{\Lambda}_{MRS} + \frac{\mu_\pi^2 - c_{cm}\left(m_h,m_b\right)\mu_G^2\left(m_b\right)}{2m_{h,MRS}} + \sum_{n=2}^{4}\frac{\rho_n}{m_{h,MRS}^n} + HMrAS\chi PT\left(m_x;a\right) + \left[\alpha_s a^2, a^{2k}, \left(am_h\right)^{2m}, \ldots\right]$$

## heavy-strange mesons



$m_{c,MRS} = 1.393\left[12\right] GeV$

(~ 0.9 %)

- a total of **67 fitting parameters** with **priors**

  prior from B*-B splitting: $\mu_G^2\left(m_b\right) = 0.35\left(7\right) GeV$

  but $\mu_G^2\left(m_b\right)$ is in the static limit

- 3 values of the lattice spacing up to $m_h \sim 1.7\ m_c$

- 2 values of the lattice spacing up to $m_h \sim 2.4\ m_c$

* it is highly desirable to perform the continuum extrapolation at each simulated heavy-quark mass **separately** with at least **3 values of the lattice spacing**

**results**

$$m_{b,MRS} = 4.751\left(13\right)_{stat}\left(12\right)_{syst}\left[18\right] GeV$$

(~ 0.4 %)

$$\bar{\Lambda}_{MRS} = 0.552\left(25\right)_{stat}\left(17\right)_{syst}\left[30\right] GeV$$

$$\mu_\pi^2 = 0.06\left(16\right)_{stat}\left(15\right)_{syst}\left[22\right] GeV$$

$\mu_\pi^2$ is still plagued by a IR renormalon ambiguity of order $\Lambda_{QCD}^2$

# CONCLUSIONS

* lattice determinations of charm and bottom quark masses have reached an impressive level of accuracy ( ~ 0.5 - 1 %)

* the treatment of discretization effects is however a delicate point at the b-quark mass

* the role of a working group like FLAG is crucial for assuring the quality of the lattice results for both the lattice and the flavor-physics communities

* besides heavy-quark masses other hadronic quantities, which are relevant in the HQE of the inclusive decay rates of heavy hadrons, are now determined with good accuracy by lattice QCD simulations

* using precise lattice data for the PS and V heavy-meson masses the matrix elements of operators of dimension-4, $\overline{\Lambda}$, dimension-5, $\mu_\pi^2$ and $\mu_G^2$, and estimates for dimension-6, $\varrho^3$, have been determined using ETMC or MILC gauge ensembles with **$N_f$ = 2+1+1 dynamical quarks** in different renormalon subtraction schemes

**kinetic scheme**

$$\begin{aligned} \overline{\Lambda} &= 0.552(26)\ GeV \\ \mu_\pi^2 &= 0.321(32)\ GeV^2 \\ \mu_G^2(\tilde{m}_b) &= 0.253(25)\ GeV^2 \\ \rho_D^3 - \rho_{\pi\pi}^3 - \rho_S^3 &= 0.153(34)\ GeV^3 \\ \rho_{\pi G}^3 + \rho_A^3 - \rho_{LS}^3 &= -0.158(84)\ GeV^3 \end{aligned}$$

**MRS scheme**

$$\overline{\Lambda} = 0.552(30)\, GeV$$