Preliminary considerations on the expansion by regions for μ-e scattering

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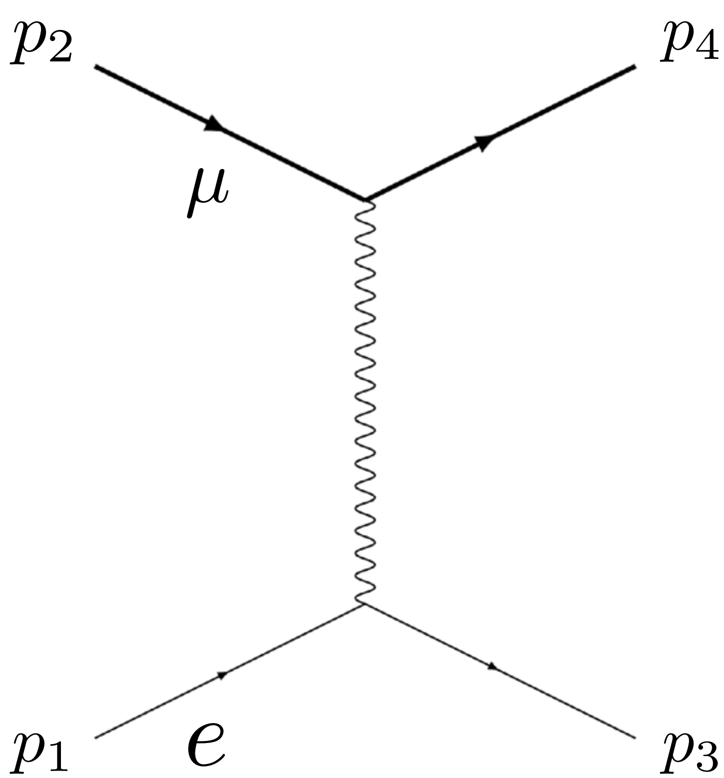
Outline

- Introduction: kinematics and counting
- Example: expansion by regions of a scalar virtual diagram
- Small mass limit
- Open questions & Summary

Kinematics and counting



$$m_\mu \sim 105\,\text{MeV}$$
$$m_e \sim 0.5\,\text{MeV}$$

Fixed target experiment frame

$$p_1 = (m_e, \vec{0})$$
$$p_2 = \left(\sqrt{m_\mu^2 + |\vec{p}_2|^2}, \vec{p}_2\right)$$

Invariants

$$s = (p_1 + p_2)^2$$
$$t = (p_1 - p_3)^2$$
$$u = (p_1 - p_4)^2$$

$$s = m_e^2 + m_\mu^2 + 2m_e \underbrace{\sqrt{m_\mu^2 + |\vec{p}_2|^2}}_{\sim 150\,\text{GeV}} \quad \to \quad \sqrt{s} \sim 400\,\text{MeV}$$

It follows that $\frac{m_\mu}{\sqrt{s}} \sim 0.25\,, \quad \frac{m_e}{\sqrt{s}} \sim 0.00125$ → $s \sim t \sim m_\mu \gg m_e$

hard scales collinear scale

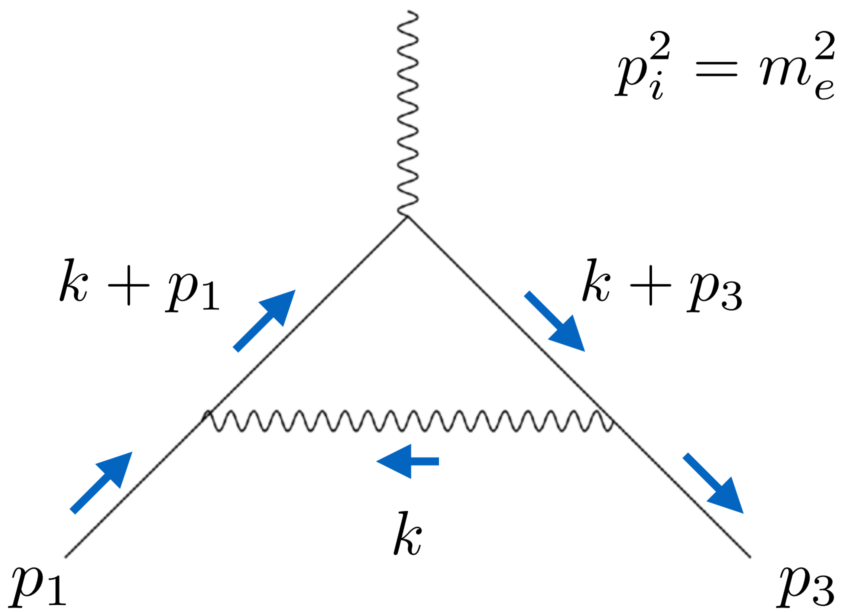
$\ln\left(s/m_\mu^2\right) \simeq 3$ $\boxed{L \equiv \ln\left(s/m_e^2\right) \simeq 14}$ $\alpha L \simeq 0.1$ → $L \sim 1/\sqrt{\alpha}$

NLO corrections

Suggested counting, different from standard QCD counting

Expansion by regions example

We focus on the electron part since m_μ is of the order of the hard scale



Expansion parameter for the scalar vertex $\lambda \equiv \frac{m_e^2}{-t} = \frac{m_e^2}{Q^2}$

$$I = i\pi^{-\frac{d}{2}}\mu^{4-d}\int d^dk \, \frac{1}{k^2\left[(k+p_1)^2 - m_e^2\right]\left[(k+p_3)^2 - m_e^2\right]}$$

After Feynman parametrisation and loop integration I obtain

$$I = \left(\frac{\mu^2}{-t}\right)^{\epsilon}\left(\frac{1}{-t}\right)\int_0^1 dx \int_0^x dy \, \frac{\Gamma(1+\epsilon)}{[-y^2 + \lambda x^2 + xy]^{1+\epsilon}}$$

After integration over the Feynman parameters and expansions

$$I = \frac{\Gamma(1+\epsilon)}{-t}\left(\frac{\mu^2}{-t}\right)^{\epsilon}\left[\frac{\ln\lambda}{\epsilon} - \frac{\pi^2}{6} - \frac{\ln^2\lambda}{2} + \mathcal{O}(\epsilon) + \mathcal{O}(\lambda)\right]$$

Now we should find/calculate the different regions that contribute to this integral

Expansion by regions example

Light-cone components $n_\mu = (1,0,0,1)$ and $\bar{n}_\mu = (1,0,0,-1)$

$$p^\mu = (\underbrace{n\cdot p}_{\text{“+ comp.”}}, \underbrace{\bar{n}\cdot p}_{\text{“− comp.”}}, p_\perp^\mu)$$

Scalar products $p\cdot q = p_+\cdot q_- + p_-\cdot q_+ + p_\perp\cdot q_\perp$

External momenta scaling $p_3 \sim (\lambda, 1, \sqrt{\lambda})Q\,, \quad p_1 \sim (1, \lambda, \sqrt{\lambda})Q$

hard-collinear scaling anti-hard-collinear scaling

Hard Region $k \sim (1,1,1)Q$

Expanded Propagators

$$k^2 \to \mathcal{O}(1)$$

$$(k+p_1)^2 - m_e^2 = k^2 + 2k_-\cdot p_{1+} + \mathcal{O}(\lambda)$$

$$(k+p_3)^2 - m_e^2 = k^2 + 2k_+\cdot p_{3-} + \mathcal{O}(\lambda)$$

$$I_h = \frac{\Gamma(1+\epsilon)}{-t}\left(\frac{\mu^2}{-t}\right)^\epsilon\left[\frac{1}{\epsilon^2} - \frac{\pi^2}{6}\right]$$

Single scale integral

Expansion by regions example

Anti-h-collinear Region $k \sim (1, \lambda, \sqrt{\lambda})Q$

Expanded Propagators

$$k^2 \sim \mathcal{O}(\lambda\ )$$

$$(k+p_1)^2 + m_e^2 = k^2 + 2k \cdot p_1 \sim \mathcal{O}(\lambda\ )$$

$$(k+p_3)^2 + m_e^2 = k^2 + 2k \cdot p_3 = 2k_+ \cdot p_{3-} + \mathcal{O}(\lambda\ )$$

$$I_c = i\pi^{-\frac{d}{2}}\mu^{4-d} \int d^d k \frac{1}{k^2[k^2 + 2k \cdot p_1][2k_+ \cdot p_{3-}]}$$

$$I_c = \frac{\Gamma(1+\epsilon)}{(2p_{1+}p_{3-})} \left(\frac{\mu^2}{m_e^2}\right)^{\epsilon} \left[-\frac{1}{2\epsilon^2} + \mathcal{O}(\epsilon) \right] =$$

Single scale integral

$$= \frac{\Gamma(1+\epsilon)}{-t} \left(\frac{\mu^2}{-t}\right)^{\epsilon} \left[-\frac{1}{2\epsilon^2} + \frac{\ln\lambda}{2\epsilon} - \frac{\ln^2\lambda}{4} + \mathcal{O}(\epsilon) + (O)(\lambda) \right]$$

integral has been rewritten

The h-collinear region $k \sim (\lambda, 1, \sqrt{\lambda})Q$ gives the same contribution as the anti-collinear region

Expansion by regions example

Sum hard, h-collinear and anti-h-collinear regions

$$I_h + 2I_c = \frac{\Gamma(1+\epsilon)}{-t}\left(\frac{\mu^2}{-t}\right)^{\epsilon}\left[\frac{1}{\epsilon^2} - \frac{\pi^2}{6} + 2\left(-\frac{1}{2\epsilon^2} + \frac{\ln\lambda}{2\epsilon} - \frac{\ln^2\lambda}{4}\right)\right] =$$

$$= \frac{\Gamma(1+\epsilon)}{-t}\left(\frac{\mu^2}{-t}\right)^{\epsilon}\left[\frac{\ln\lambda}{\epsilon} - \frac{\pi^2}{6} - \frac{\ln^2\lambda}{2}\right] = I$$

The sum of these 3 regions reproduces the initial integral, this proves that other regions do not contribute, for example (ultra-)soft regions must give scaleless integrals

Soft Region $k \sim (\lambda, \lambda, \lambda)Q$

$$k^2 \sim \mathcal{O}(\lambda^2)$$
$$(k+p_1)^2 - m_e^2 = 2k_- \cdot p_{1+} + \mathcal{O}(\lambda^2)$$
$$(k+p_3)^2 - m_e^2 = 2k_+ \cdot p_{3-} + \mathcal{O}(\lambda^2)$$

$\longrightarrow$ $I_{s,\,us} = 0$

It is possible to prove that this integral vanishes in dimensional regularisation, the integral in the ultra-soft region $k \sim (\lambda^2, \lambda^2, \lambda^2)Q$ is the same as the one in the soft region (but with a different scaling) and it also vanishes

Factorization

The expansion by regions is important to find a factorization theorem (separation of scales)

$$d\sigma \sim H(s,t,m_\mu,\mu)F_j(m_e,\mu)F_j(m_e,\mu)\ldots$$

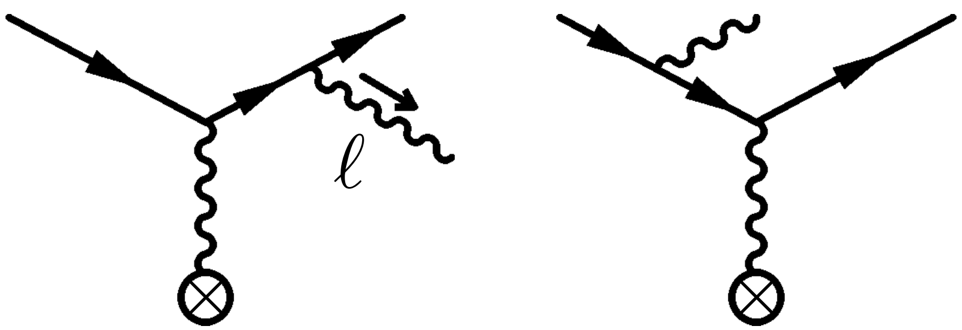
hard scales   collinear scale

Missing terms/regions

- Single scale objects, then resummation is possible via RG-evolution
- We need to look at the real emission diagrams, new scales and regions (usually) appear: soft, soft-collinear (?)
- Question? Is there an experimental soft cutoff in the energy of the emitted photons? I think in practice there is one: “The angles of the scattered electron and muon are correlated…This constraint is extremely important to select elastic scattering events, rejecting background events from radiative or inelastic processes”
- We call this soft photon cutoff ΔE. What is the size of this scale ΔE? $m_e \ll \Delta E \ll m_\mu$, s,t or $m_e \sim \Delta E$ (probably not)?

Regions in the real emission diagrams

Soft real emission from static source, Moeller scattering, (R. Hill, [arXiv:1605.02613]) in HQET then expanded in $m_e^2 << Q^2$, with soft cutoff on the photon energy



Soft function (IR subtracted)

$$S^{(1)} = -4\left(\log\frac{\mu^2}{m^2} + \log\frac{E_e^2}{(\Delta E)^2}\right)(L-1) + 2L^2 + 4\mathrm{Li}_2\left(\cos^2\frac{\theta}{2}\right) - \frac{4\pi^2}{3}$$

Large logarithms still present :-(

- Large logarithmic corrections are still present in this formula, further separation of regions is needed

- It seems to be a situation similar to the boosted heavy quark regime: need to study the enhanced $\ln(m_e^2/Q^2)$ contributions in the soft-emission limit $\Delta E^2 \ll Q^2$ at fixed $(\Delta E)^2/E_e^2$ (joint limit, the two limits are independent)

Regions in the real emission diagrams

Soft real emission from a static source, expansion (of one of the relevant integrals) in m_e/E

$$I_r = \int_{k^0 \leq \Delta E} \frac{d^{d-1}k}{(2\pi)^{d-1} 2k^0} \frac{-2p \cdot p'}{(p \cdot k)(p' \cdot k)}$$

Soft Region

$$I_{r,s} = -\frac{1}{\epsilon^2} + \frac{2\ln\left[\frac{\Delta E}{\mu}\right]}{\epsilon} + \frac{\pi^2}{4} - 2\ln^2\left[\frac{\Delta E}{\mu}\right]$$

Collinear Region

$$I_{r,c} = \frac{1}{2\epsilon^2} - \frac{\ln\left[\frac{m_e \Delta E}{E\mu}\right]}{\epsilon} + \frac{\pi^2}{24} + \ln^2\left[\frac{m_e \Delta E}{E\mu}\right]$$

New soft-collinear scale arises

$$\frac{m_e \Delta E}{E}$$

Sum soft, collinear and anti-collinear regions

$$I_{r,s} + I_{r,c} + I_{r,\bar{c}} = -\frac{\ln\left[\frac{m_e^2}{E^2}\right]}{\epsilon} + \frac{1}{2}\ln^2\left[\frac{m_e^2}{E^2}\right] + \ln\left[\frac{m_e^2}{E^2}\right]\ln\left[\frac{\Delta E^2}{\mu^2}\right] + \frac{\pi^2}{3}$$

Scale separation has now been achieved, a soft-collinear scale is present in the calculation

Small mass limit & factorization

Mele Nason '91, Melnikov, Arbuzov '02, Melnikov Mitov '04, Mitov Moch '07,….

$$\frac{d\sigma_{\mathcal{Q}}}{dz}(z,Q,m) = \sum_a \int_z^1 \frac{dx}{x}\frac{d\hat{\sigma}_a}{dx}(x,Q,\mu)D_{a/\mathcal{Q}}\left(\frac{z}{x},\frac{\mu}{m}\right)$$

Cross section for the production of a massless parton a

Fragmentation function: probability that a massless parton fragments into a massive quark. Describes collinear radiation to final-state particles

Similar to the simple example of the expansion by regions above

Now expand in the soft limit, further factorization (limits should be independent and commutative?)

$$D(z,m_e,\mu) = F_j(m_e,\mu)S_j(m_e\Delta E/E,\mu) + \mathcal{O}(\Delta E/E_e)$$

Korchemsky Marchesini '93, Cacciari Catani '01, Gardi '05, Neubert '07, Ferroglia Pecjak Yang '12

Naive/Guess factorization theorem

$$d\sigma \sim H(s,t,m_\mu,m_e=0,\mu)F_j(m_e,\mu)F_j(m_e,\mu)S(\Delta E,s,t,m_\mu,m_e=0,\mu)S_{j,i}(m_e\Delta E/E_e,\mu)S_{j,f}(m_e\Delta E/E_e,\mu)$$

Hard function, virtual corrections with m_e=0

virtual corrections of the fragmentation functions

Soft function, could contain ratio of hard scales

Soft-collinear functions for initial and final state e

$$+\mathcal{O}(\Delta E/E_e) + \mathcal{O}(m_e^2/s)$$

Resummation

In some convenient space (momentum or Laplace/Mellin…) the functions appearing in the factorization formula satisfy RG equations of the type

RG equation

$$\frac{d}{d\ln\mu} H(Q^2,\mu) = 2\left[\gamma_{\rm cusp}\ln\frac{Q^2}{\mu^2} + \gamma(\alpha)\right] H(Q^2,\mu)$$

Formal solution

$$H(Q^2,\mu) = \exp\left\{\int_{\mu_h}^{\mu} 2\left[\gamma_{\rm cusp}\ln\frac{Q^2}{\mu^2} + \gamma(\alpha)\right] d\ln\mu'\right\} H(Q^2,\mu_h)$$

Change of variables, QED running coupling is needed

$$\frac{d\alpha(\mu)}{d\ln\mu} = \beta(\alpha(\mu)) \quad \longrightarrow \quad \ln\frac{\nu}{\mu} = \int_{\alpha(\mu)}^{\alpha(\nu)} \frac{d\alpha}{\beta(\alpha)}$$

$$\log\left(\frac{H(\mu_L)}{H(\mu_H)}\right) = \underbrace{-\frac{\gamma_0}{\beta_0}\left\{\log r + \dots\right\}}_{\alpha^{1/2}} \underbrace{-\frac{\gamma_0^{\rm cusp}}{\beta_0}\left\{\log\frac{Q^2}{\mu_H^2}\log r + \frac{1}{\beta_0}\left[\frac{4\pi}{\alpha(\mu_H)}\left(\frac{1}{r} - 1 + \log r\right)\right.\right.}_{\alpha^0}$$

$$\underbrace{\left.\left. + \left(\frac{\gamma_1^{\rm cusp}}{\gamma_0^{\rm cusp}} - \frac{\beta_1}{\beta_0}\right)(-\log r + r - 1) - \frac{\beta_1}{2\beta_0}\log^2 r\right]}_{\alpha} + \dots\right\} \qquad r = \alpha(\mu_L)/\alpha(\mu_H)$$

[arXiv:1605.02613]

Different counting than QCD

What is needed

- Find all the relevant hard, collinear and soft kinematic scales of the process (m_μ,s,t,E_e,m_e,ΔE,$m_e\Delta E/E_e$,...), be careful with hidden low energy scales that could possibly be introduced experimentally (ΔE). How do ΔE and m_e relate to each other?

- Formally prove factorization formula by employing effective field theory methods

- Explicit computation of the coefficients entering the factorization formula (at NLO first and eventually at NNLO)

- Fixed order results from factorization theorem where power corrections in m_e^2/Q^2 and $\Delta E/E_e$ are neglected

- Resummation by RG evolution (it directly depends on the structure of the factorization theorem)

Thank you!