The Evaluation of the Leading Hadronic Contribution to the Muon Anomalous Magnetic Moment

Muon decay at NNLO
Status Update

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disclaimer

- this is meant to be a “workshop” talk
  (not a polished conference talk, everything is in flux)
- a status update about the 2-loop muon decay $\mu \to e + (\nu\bar{\nu})$
  (heavy-to-light form factor)
- simpler than $\mu\, e$ scattering but with overlap
  (multiple separated mass scales: $m \ll M \sim s$)
- useful environment to learn for $\mu\, e$ scattering

landscape

we assume:

- the LO $\mu\, e$ scattering is known with full $m \equiv m_e$ dependence
- the (fully differential) NLO $\mu\, e$ scattering is known with full m dependence.
  - is there any value in another (independent) Monte Carlo?
  - to cross check, resummation of $\ln(m/M)$ . . .
- "someone" computes two-loop amplitude for $\mu\, e$ scattering with $m = 0$
- nobody can do the master integrals with $m \neq 0$

we want to provide:

- fully differential Monte Carlo for $\mu\, e$ scattering up to 'NNLO'
- drop terms suppressed by $\alpha^2 z \equiv \alpha^2 m/M$ (relative to LO)
- but keep αz^n and $\alpha^2 (\ln z)^n$

landscape

we need to:

- construct two-loop matrix element $m=0 \;\to m\neq 0$ (not full m dependence, only $\ln z$ dependence)
  - is this possible ? $\Rightarrow$ Yes
  - take m as collinear regulator (works in QED)
  - collinear singularities (i.e $\ln m$ terms) have universal structure
- use (FKS) subtraction at NNLO to deal with double-real and real-virtual IR (soft) phase-space singularities
- match expansion in z between real and virtual (e.g. for one-loop $\mu e \to \mu e \gamma$)

test case: $\mu \to e + (\nu\bar{\nu})$

- simplest process with two different non-vanishing masses
- produce fully differential 'NNLO' Monte Carlo

test case: μ decay

Calculate muon (and maybe top) decay fully differential at NNLO with:

$0 < m \ll M \sim s$

- three scales (M, m and s) $\Rightarrow$
  integrals recently computed [Chen,1801.01033]
  ideal for testing expansion/factorization
  - expand $\mathrm{d}\Gamma_{\mathrm{VV}}$ in $z = m/M$ and drop terms $\mathcal{O}(z)$
  - derive two-mass fragmentation function (to be applied in $\mu\, e$ scattering)
- 'done' for e energy spectrum [Arbuzov, Melnikov 02]
- (scheme dependence of $\mathrm{d}\Gamma_{\mathrm{VV}}$ (known), $\mathrm{d}\Gamma_{\mathrm{RV}}$ and $\mathrm{d}\Gamma_{\mathrm{RR}}$, play with γ_5)

$m = 0 \;\rightarrow\; m \neq 0$

Use SCET inspired way to relate $m = 0 \;\rightarrow\; m \neq 0$ [Becher, Melnikov 07]

($\sim$ fragmentation function approach)

Form factor: (only one external mass $m \ll Q^2 = s$)

$$F(s, m, \{m_i^2\}) = Z_J(m^2, \{m_i^2\})\, S(s, \{m_i^2\})\, \tilde{F}(s) + \mathcal{O}(m^2/s)$$

- $S(s, \{m_i^2\})$: soft function, only contributions from vacuum polarization diagrams with massive fermions, $\supset \ln(m^2/s)$
- $Z_J(m^2)$: jet fct., independent of hard scale s, $\supset \ln(m^2/m_i^2)$
- $\tilde{F}(s)$: massless form factor
- factorisation $\leftrightarrow$ resummation via RG equations

Bhabha scattering:

$$M(\{p_i\}, \{m_i^2\}) = Z_J^2(m^2)\, S(s, t, u, \{m_i^2\})\, \tilde{M}(\{p_i\}) + \mathcal{O}(m^2/\{s, t, u\})$$

$m = 0 \;\to\; m \neq 0$

- Extend this to processes with two external masses, M and m
- simplest example: $\mu(p) \to e(q) + (\nu\bar{\nu})$
- Kinematics: $p^2 = M^2,\; q^2 = m^2$
- Write $p = p_+ + p_-$, $q = q_- + q_\perp$

  $\Rightarrow M^2 = p^2 = 2p_+ \cdot p_-$, $m^2 = q^2 = q_\perp^2$, $s = 2p_+ \cdot q_-$
- $q = (q_+, q_-, q_\perp) \sim (0, 1, \lambda)$ and $p \sim (1, 1, 0)$

  with $\lambda \sim m/M \ll 1$
- use method of regions (MoR): expand integrand
  - hard, h: $k \sim (1, 1, 1)$
  - soft, s: $k \sim (\lambda, \lambda, \lambda)$
  - collinear, c: $k \sim (\lambda^2, 1, \lambda)$
  - ultrasoft, us: $k \sim (\lambda^2, \lambda^2, \lambda^2)$
  - anti-collinear, $\bar{c}$: $k \sim (1, \lambda^2, \lambda)$

MoR: simple example

- $\mathcal{I}_{111} = \int_{k_1\,k_2} ((k_1-k_2)^2)^{-1}(k_2^2 - 2k_2\cdot q)^{-1}(k_1^2 - 2k_1\cdot p)^{-1}$
- only hard and collinear contributions
- $\mathcal{I}_{111}^{h_1\text{-}h_2}$ naive polynomial expansion in m^2, $\mathcal{I}_{111}^{h_1\text{-}c_2}$ contains $\ln(z)$

$$\mathcal{I}_{111}^{h_1\text{-}h_2} = \int_{k_1\,k_2} \frac{1}{(k_1-k_2)^2}\frac{1}{k_1^2-2k_1\cdot p} \times\left(\frac{1}{k_2^2-2k_2\cdot q_-} + \lambda^2\,\frac{4(k_2\cdot q_\perp)^2}{(k_2^2-2k_2\cdot q_-)^3} + \mathcal{O}(\lambda^4)\right)$$

$$\mathcal{I}_{111}^{h_1\text{-}c_2} = \int_{k_1\,k_2} \frac{1}{k_1^2-2k_1\cdot p}\frac{1}{k_2^2-2k_2\cdot q} \times\left(\frac{1}{k_1^2-2k_1\cdot k_2^-} + \lambda^2\left[\frac{4(k_1\cdot k_2^\perp)^2}{(k_1^2-2k_1\cdot k_2^-)^3} + \frac{2k_1\cdot k_2^+ - k_2^2}{(k_1^2-2k_1\cdot k_2^-)^2}\right]\right)$$

NLO warm up

- For $\mu \to e\nu\bar{\nu}$ we have $F(s,M,m) \simeq \sqrt{Z_J(m)}\,\tilde{F}(s,M)$:

$$F^{(1)}(s,M,m) \simeq \tilde{F}^{(1)}(s,M) \underbrace{- \frac{\alpha}{4\pi} m^{-2\epsilon} \left(\frac{1}{\epsilon^2} + \frac{1}{2\epsilon} + \zeta(2) + 2 \right)}_{1/2\,\delta Z_J^{(1)}(m^2)} \tilde{F}^{(0)}(s,M)$$

- $S^{(1)}(s,m) = 1$ because there are no internal fermion loops
- only hard and collinear contribute at NLO
- For $\mu\, e$ scattering we have $\mathcal{M}(s,t,M,m) \simeq Z_J \tilde{\mathcal{M}}(s,t,M)$:

$$\mathcal{M}^{(1)}(M,m) \simeq \tilde{\mathcal{M}}^{(1)}(M) + \delta Z_J^{(1)}(m) \tilde{\mathcal{M}}^{(0)}(M)$$

  checked explicitly
- method of regions and factorization works at NLO

NNLO

$\mu(p) \to e(q) + \nu\bar{\nu}$

- Qgraf $\to$ FORM $\to$ Reduze $\to$ FORM
- reduction to 38 planar and 2 non-planar master integrals
- use projectors to obtain form factors
- master integrals 'exact' available, ongoing:
    - check IR singularities
    - check scheme dependence
    - check limit $m \to M$
- but we need expansion via MoR to
    - cross check factorization
    - obtain $\delta Z_J^{(2)}(m)$ and soft function
- $\Rightarrow$ master integrals through method of regions

MoR @ two-loop

- semi-automatic in Mathematica:

  try all regions $k_i \sim (\lambda^a, \lambda^b, \lambda^{(a+b)/2})$ with $0 \le a, b \le 4$
- in all but two cases: single Mellin-Barnes and `Integrate`
- leading term of h_1-h_2 is $\mathcal{I}(m \to 0, q \to q_-)$
- nearly all master integrals expanded using MoR (two missing)

  $\Rightarrow$ comparison against exact results
- through IBP reduction, get prefactors $1/m^2$

  $\Rightarrow$ need expansion in z higher than naively expected
- individual integrals depend on h, c, s, us, but not $\bar{c}$
- expect us to drop out of $F(s, m, M, \{m_i^2\})$
- sometimes (c-c and s-s) additional analytic regularization needed (dim reg not sufficient), cancels within a single master integral.

real correction at (N)NLO

- NLO completely solved, several times
- many general schemes for NNLO phase-space integrations
- for general QCD calculations not very simple
- our case(s) (i.e. massive QED): much simpler
- only soft$\times$soft singularities
- extend FKS to NNLO (as it treats soft singularities separately)

real correction at NLO

The FKS formalism at NLO

- no collinear singularities $\to$ very simple scheme
- Let $E_\gamma \propto \xi$ and $\cos\sphericalangle(l,\gamma) = y$, introduce arbitrary $0 < \xi_{\text{cut}} \le 1$

$$\mathrm{d}\Gamma_R = \underbrace{\mathrm{d}\phi_{n+1}}_{\propto \xi^{1-2\epsilon}\mathrm{d}\xi} \underbrace{\mathcal{M}_{n+1}}_{\supset \xi^{-2}} \propto \mathrm{d}\phi_n \times \frac{\mathrm{d}\xi\,\mathrm{d}y\,\mathrm{d}\Omega^{(2-2\epsilon)}}{(1-y^2)^\epsilon} \xi^{-1-2\epsilon} \underbrace{\left(\xi^2 \mathcal{M}_{n+1}\right)}_{\text{reg. } \xi\to 0}$$

$$\propto \left(\underbrace{-\frac{\xi_{\text{cut}}^{-2\epsilon}}{2\epsilon}\delta(\xi)}_{(s)} + \underbrace{\left(\xi^{-1-2\epsilon}\right)_{\xi_{\text{cut}}}}_{(h)} \right) \left(\xi^2 \mathcal{M}_{n+1}\right)$$

with $\int \mathrm{d}\xi \, (\xi^n)_{\xi_{\text{cut}}} f(\xi) = \int \mathrm{d}\xi \, \xi^n \Big(f(\xi) - f(0)\theta(\xi_{\text{cut}} - \xi) \Big)$

- $\mathrm{d}\Gamma_R^{(s)}$ (= soft) contains the poles and $\Rightarrow \hat{\mathcal{E}}(\xi_{\text{cut}})$
  $\mathrm{d}\Gamma_R^{(h)}$ (= hard) is finite $\to$ integrate numerically with $\epsilon \to 0$

real correction at NNLO

Extend FKS to NNLO $\to$ two terms (also applicable for $\mu\, e$)

term 1: real $\times$ real:

- iterate FKS (four terms instead of two) $\to$ (h) mixes with (s)

$$\mathrm{d}\Gamma_{RR} = \underbrace{\mathrm{d}\Gamma_{RR}^{(hh)}}_{\text{finite}} + \mathrm{d}\Gamma_{RR}^{(hs)} + \mathrm{d}\Gamma_{RR}^{(sh)} + \underbrace{\mathrm{d}\Gamma_{RR}^{(ss)}}_{\hat{\mathcal{E}}(\xi_{1,\text{cut}})\,\hat{\mathcal{E}}(\xi_{2,\text{cut}})}$$

- $\mathrm{d}\Gamma_{RR}^{(hs)}$ and $\mathrm{d}\Gamma_{RR}^{(sh)}$ introduce

$$\mathcal{I}(\xi_{1,\text{cut}}, \xi_{2,\text{cut}}) \propto \int \mathrm{d}\xi\, \mathrm{d}y\, (1-y^2)^{-\epsilon} \left(\frac{1}{\xi^{1+2\epsilon}}\right)_{\xi_{2,\text{cut}}} \left(\hat{\mathcal{E}}(\xi_{1,\text{cut}})\xi^2 \mathcal{M}_{n+1}^{(0)}(\xi, y)\right)$$

- $\hat{\mathcal{E}}$ is the integrated eikonal, has $1/\epsilon$ pole
- In all generality $\mathcal{I}(\xi_{1,\text{cut}}, \xi_{2,\text{cut}})$ is tricky, but ...

real correction at NNLO

Extend FKS to NNLO $\to$ two terms (also applicable for $\mu\, e$)

term 2: real $\times$ virtual:

- like NLO but $\mathrm{d}\Gamma^{(h)}_{RV}$ contains explicit poles $1/\epsilon$ from $\mathcal{M}^{(1)}_{n+1}$
- write $\mathcal{M}^{(1)}_{n+1} = \underbrace{\mathcal{M}^{(1,\text{fin})}_{n+1}}_{\epsilon\to 0} \underbrace{- \hat{\mathcal{E}}(\xi_\text{cut})\mathcal{M}^{(0)}_{n+1}}_{\Rightarrow\ \mathrm{d}\Gamma^\text{sin}_{RV} = -\mathcal{I}(\xi_\text{cut},\xi_\text{cut})}$
- magic:

$$\mathrm{d}\Gamma^\text{sin}_{RV} + \mathrm{d}\Gamma^{(hs)}_{RR} + \mathrm{d}\Gamma^{(sh)}_{RR}\Big|_{\xi_{i,\text{cut}}\ \text{equal}} = 0$$

- process dependent structure cancels

status/outlook

virtual:

- ✓ checked form factor using Chen's integrals
- ✓ determined jet function at one loop and applied to μe scattering
- two integrals to be calculated and determination of jet function

real$\times$virtual $+$ real$\times$real:

- Implement and test FKS subtraction at NNLO
- study expansion of real in z to match with virtual

compare and combine with other approaches:

- would be happy to organize a workshop