

Building bases for analytical fits of four-loop master integrals

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Evaluation of the Leading Hadronic Contribution to a_μ

MITP Mainz,

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Summary

- 4-loop $g-2$ summary
- the master integrals of 4-loop $g-2$
- Families of constants
- Polylogarithms
- Elliptical constant

Mass-independent QED contribution, 1-3 loop

see also Marc's talk

$$a_e^{QED} = C_1 \left(\frac{\alpha}{\pi}\right) + C_2 \left(\frac{\alpha}{\pi}\right)^2 + C_3 \left(\frac{\alpha}{\pi}\right)^3 + C_4 \left(\frac{\alpha}{\pi}\right)^4 + C_5 \left(\frac{\alpha}{\pi}\right)^5 + \dots$$

$$C_1 = \frac{1}{2} \quad (\text{Schwinger 1948}) \quad 1 \text{ diagram} \quad \img alt="Feynman diagram for C1: a single fermion loop with a photon insertion." data-bbox="555 275 630 345"/>$$

$$C_2 = \frac{197}{144} + \frac{1}{12}\pi^2 - \frac{1}{2}\pi^2 \ln 2 + \frac{3}{4}\zeta(3) \quad \img alt="Feynman diagrams for C2: seven diagrams showing two-loop corrections to the photon insertion." data-bbox="490 385 925 460"/>$$

$$= -0.328\,478\,965\,579\dots, \quad (\text{Petermann, Sommerfeld 1957}) \quad 7 \text{ diagrams}$$

$$C_3 = \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5) + \frac{100}{3} \left[\left(a_4 + \frac{1}{24}\ln^4 2 \right) - \frac{1}{24}\pi^2\ln^2 2 \right] \quad \img alt="Feynman diagrams for C3: three diagrams showing three-loop corrections." data-bbox="820 570 895 645"/>$$

$$- \frac{239}{2160}\pi^4 + \frac{139}{18}\zeta(3) - \frac{298}{9}\pi^2\ln 2 + \frac{17101}{810}\pi^2 + \frac{28259}{5184} \quad \img alt="Feynman diagrams for C3: two diagrams showing three-loop corrections." data-bbox="820 675 895 750"/>$$

$$= 1.181\,241\,456\dots, \quad (\text{S.L., E.Remiddi 1996}) \quad 72 \text{ diagrams} \quad \img alt="Feynman diagram for C3: one diagram showing a three-loop correction." data-bbox="820 775 895 850"/>$$

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}, \quad a_4 = \sum_{n=0}^{\infty} \frac{1}{2^n n^4}, \quad \dots$$

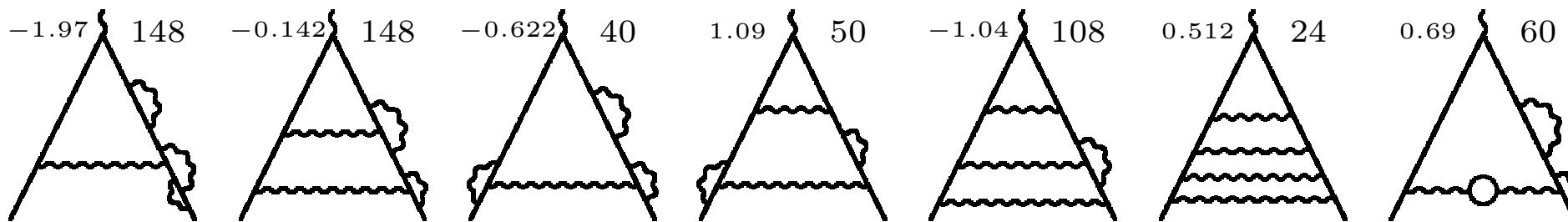
Mass-independent QED contribution, 4-loop

$C_4 \rightarrow$ **891 diagrams** obtained by inserting a photon in 104 self-mass diagrams
1100-digits result is (S.L., 2017):

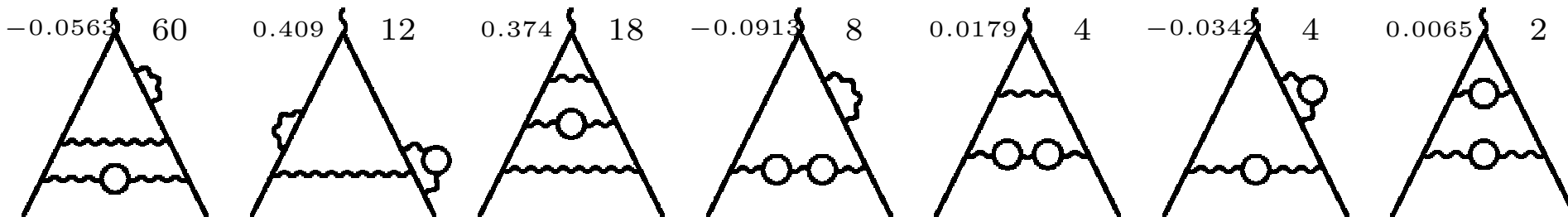
$C_4 =$
-1.9122457649264455741526471674398300540608733906587253451713298480060
3844398065170614276089270000363158375584153314732700563785149128545391
9028043270502738223043455789570455627293099412966997602777822115784720
3390641519081665270979708674381150121551479722743221642734319279759586
0740500578373849607018743283140248380251922494607422985589304635061404
9225266343109442400023563568812806206454940132249775943004292888367617
4889923691518087808698970526357853375377696411702453619601349757449436
1268486175162606832387186747303831505962741878015305514879400536977798
3694642786843269184311758895811597435669504330483490736134265864995311
6387811743475385423488364085584441882237217456706871041823307430517443
0557394596117155085896114899526126606124699407311840392747234002346496
9531735482584817998224097373710773657404645135211230912425281111372153
0215445372101481112115984897088422327987972048420144512282845151658523
6561786594592600991733031721302865467212345340500349104700728924487200
6160442613254490690004319151982300474881814943110384953782994062967586
787538524978194698979313216219797575067670114290489796208505...

- For the sake of comparisons it is convenient to regroup the 891 diagrams in 25 gauge-invariant sets
- The contribution of gauge-invariant sets are **I.R. finite** and invariant under gauge transformations of the internal photons.

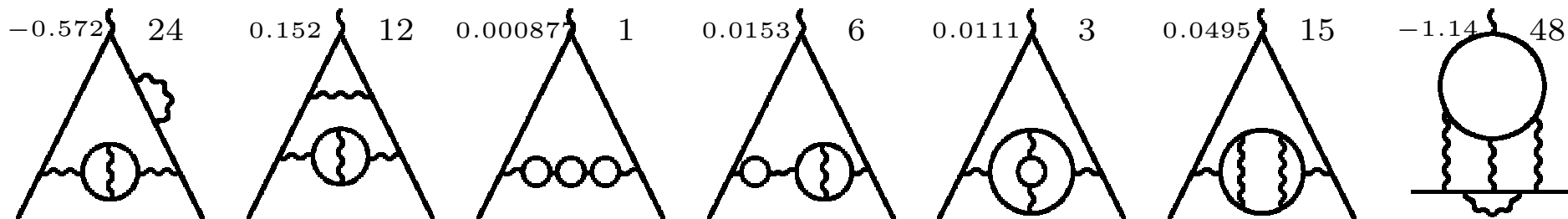
one diagram of each one of the 25 gauge-invariant sets



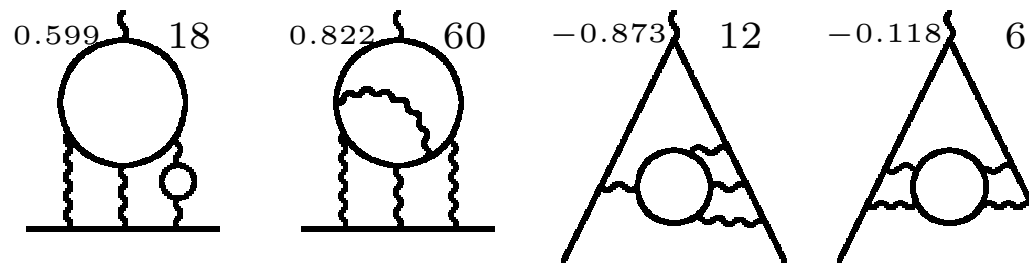
(1) (2) (3) (4) (5) (6) (7)



(8) (9) (10) (11) (12) (13) (14)



(15) (16) (17) (18) (19) (20) (21)



(22) (23) (24) (25)

Typical representative diagrams of gauge-invariant sets

Left: contribution to $g-2$ right: number of diagrams

Steps for $g-2$

- reduction (exact) of the contribution to 334 master integrals
- high-precision numerical calculation of master integrals
- classification of master integrals \Leftarrow
- fit to an analytical basis

A posteriori the analytical expressions of the master integrals turn out to contain elements of different families of analytical constants:

- harmonic polylogarithms of 1 and $1/2$.
- harmonic polylogarithms of arguments $e^{i\pi/3}$ and $e^{2i\pi/3}$
- harmonic polylogarithms of arguments $e^{i\pi/2}$
- elliptic constants with semi-analytic representation
- unknown elliptic constants

up to weight 7

See the colors in the next slide

the 104 4-loop electron self-masses



$HPL(e^{i\pi/3})$ elliptic $HPL(e^{i\pi/3}) +$ elliptic
 $HPL(e^{i\pi/2}) + HPL(e^{i\pi/3})$ $HPL(e^{i\pi/2}) + HPL(e^{i\pi/3}) +$ elliptic

Steps for $g-2$

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Recipe for analytical fits with PSLQ

1. compute an extremely high-precision value of a master integral
2. guess the right analytical ansatz for the basis
3. fit an analytical expression by using the “PSLQ algorithm”
 - [PSLQ algorithm](#) (Ferguson and Bailey 1992)
 - multi-integers extension of the Euclid algorithm for the calculation of the GCD of two integers
 - it finds an integer relation between floating point numbers or bounds on size of coefficients.
 - it requires **high precision**; at least number of digits of coefficients * number of real numbers
 - there are various implementations of the algorithm publicly available; the fastest is a parallel modification of the algorithm by Bailey and Broadhurst 1999.

The simplest family: Harmonic polylogarithms of 1, 1/2

weight	number	constants
0	1	1
1	1	$\ln 2$
2	2	$\zeta(2), \ln^2 2$
3	3	$\zeta(3), \zeta(2) \ln 2, \ln^3 2$
4	5	$\zeta(4), \zeta(3) \ln 2, \zeta(2) \ln^2 2, \ln^4 2, a_4$
5	8	$\zeta(5), \zeta(4) \ln 2, \zeta(3) \ln^2 2, \zeta(2) \ln^3 2, \zeta(3)\zeta(2), a_4 \ln 2, \ln^5 2, a_5$
6	13	$\zeta(6), \zeta(3)^2, \dots, a_5 \ln 2, \ln^6 2, a_6, b_6$
7	21	$\zeta(7), \zeta(4)\zeta(3), \dots, b_6 \ln 2, \ln^7 2, a_7, b_7, d_7$
n	F_{n+1}	

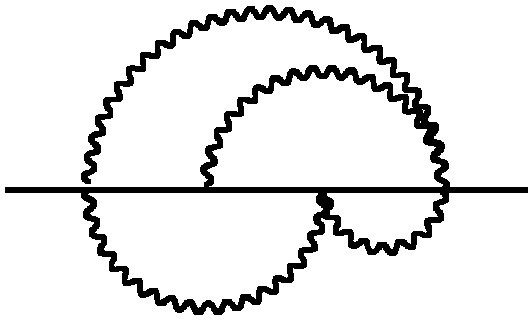
Fibonacci numbers $F_0 = 1, F_1 = 1, F_{n+1} = F_n + F_{n-1}$,

$\zeta(n) = \text{Li}_n(1), a_n = \text{Li}_n(1/2)$ common polylogarithms

$b_6 = S_{4,2}(1/2), b_7 = S_{5,2}(1/2)$ Nielsen polylogarithms

$d_7 = H_{0,0,0,0,1,-1,-1}(1)$ only with harmonic polylogarithms

PSLQ example: analytical fit of a simple master integral



$$M = G_1 \epsilon^{-4} + G_2 \epsilon^{-3} + G_3 \epsilon^{-2} + G_4 \epsilon^{-1} + G_5 + G_6 \epsilon + G_7 \epsilon^2 + G_8 \epsilon^3 + \dots$$

$$\epsilon = (4 - D)/2$$

G_i calculated numerically and fitted

$$G_1 = -\frac{1}{8}$$

$$G_2 = -\frac{49}{48}$$

$$G_3 = -\frac{449}{96} - \frac{1}{6}\pi^2 - \frac{1}{2}\zeta(3)$$

$$G_4 = -\frac{2429}{192} - \frac{7}{4}\pi^2 - \frac{11}{2}\zeta(3) - \frac{1}{120}\pi^4$$

$$G_5 = \frac{2687}{384} - \frac{277}{24}\pi^2 - \frac{125}{3}\zeta(3) + \frac{1}{24}\pi^4 + \frac{3}{2}\zeta(5) + \frac{2}{3}\pi^2\zeta(3)$$

$$G_6 = \frac{95689}{256} - \frac{2377}{48}\pi^2 - \frac{2999}{12}\zeta(3) - \frac{46}{45}\pi^4 + \frac{149}{2}\zeta(5) + \frac{15}{2}\zeta(3)^2 - \frac{58}{3}\pi^2\zeta(3) - \frac{29}{270}\pi^6$$

PSLQ example: analytical fit of a simple master integral

$$\begin{aligned}
 G_7 = & -2342.207514106023075423522540590792709885328732056559470807 \\
 & 359481483571384691680645591697318599261483194890419734356986 \\
 & 640536482839180927737599376306979737829110608311707671767935 \\
 & 983139125960766918329923883871930584868496516072868729243183 \\
 & 317800519694759939914751761141283435810030791136838793708071 \\
 & 157346099787020302357526852412095436287332846448926242430503 \\
 & 236449547474407307581291123637921078586418676517549877972867
 \end{aligned}$$

.....

$$\begin{aligned}
 = & \frac{1671597}{512} - \frac{4381}{96} \pi^2 - \frac{22193}{24} \zeta(3) - 144 \pi^2 \ln 2 - \frac{3617}{240} \pi^4 - \frac{71}{2} \zeta(5) \\
 & - \frac{393}{2} \pi^2 \zeta(3) - \frac{869}{162} \pi^6 - 24 \pi^4 \ln^2 2 + 576 \pi^2 a_4 + 24 \pi^2 \ln^4 2 - \frac{803}{2} \zeta(3)^2 \\
 & + 504 \pi^2 \zeta(3) \ln 2 - \frac{1735}{4} \zeta(7) + \frac{799}{6} \pi^2 \zeta(5) - \frac{661}{180} \pi^4 \zeta(3)
 \end{aligned}$$

black: ansatz brown: PSLQ result

example output pslq

```
PSLQM3 integer relation detection: n = 54
Iteration 532 updtmp: Min, max of y = 7.125852D -124 8.335992D -121
Iteration 532 Norm bound = 1.330171D 1 Max. bound = 1.330171D 1
Iteration 1210 updtmp: Min, max of y = 4.109989D -247 8.231208D -243
Iteration 1210 Norm bound = 2.868041D 3 Max. bound = 2.868041D 3
Iteration 1887 updtmp: Min, max of y = 2.529002D -368 5.851086D -365
Iteration 1887 Norm bound = 5.545033D 5 Max. bound = 5.545033D 5
Iteration 2174 updtmp: Min, max of y = 1.427865D -1193 4.367510D -415
Iteration 2174 updtmp: Small value in y = 1.427865D -1193
Iteration 2174 Norm bound = 4.643888D 6 Max. bound = 4.643888D 6
Iteration 2174 Relation detected
Min, max of y = 1.427865D -1193 4.367510D -415
Max. bound = 4.643888D 6
Index of relation = 5 Norm = 1.26788D 7 Residual = 1.427865D -1193
CPU times:
    0.17    0.02    2.88    2.28    5.75    0.00
Recovered relation: 0 =
+
+          -5014791. * 1
+          420576. * z2
+          1420352. * z3
+          1327104. * z2*lg2
+          2083392. * z4
+          54528. * z5
+          1810944. * z3*z2
+          7786240. * z6
+          3317760. * z4*lg2^2
+          -221184. * z2*lg2^4
+          616704. * z3^2
+          -4644864. * z3*z2*lg2
+          -5308416. * a4*z2
+          666240. * z7
+          -1227264. * z2*z5
+          507648. * z4*z3
+          1536. * G7
CPU Time = 13.3480
```

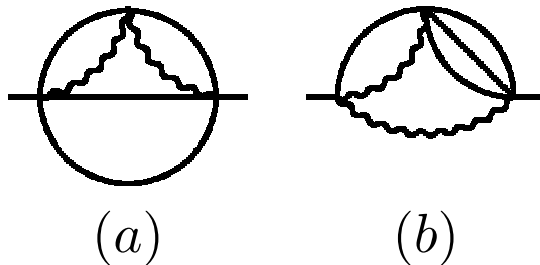
- Internal implementation in Mathematica
- C code from Paul Zimmermann
<https://members.loria.fr/PZimmermann/software/pslq-1.0.c>
- Multiprecision fortran code from David Bailey, implementing sequential and parallel PSLQ <http://crd-legacy.lbl.gov/~dhbailey/mpdist/>

C₄ analytical fit part 1

$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U \quad 46/54 \text{ terms}$$

$$\begin{aligned}
 T = & \frac{1243127611}{130636800} + \frac{30180451}{25920} \zeta(2) - \frac{255842141}{2721600} \zeta(3) - \frac{8873}{3} \zeta(2) \ln 2 + \frac{6768227}{2160} \zeta(4) \\
 & + \frac{19063}{360} \zeta(2) \ln^2 2 + \frac{12097}{90} \left(a_4 + \frac{1}{24} \ln^4 2 \right) - \frac{2862857}{6480} \zeta(5) - \frac{12720907}{64800} \zeta(3) \zeta(2) \\
 & - \frac{221581}{2160} \zeta(4) \ln 2 + \frac{9656}{27} \left(a_5 + \frac{1}{12} \zeta(2) \ln^3 2 - \frac{1}{120} \ln^5 2 \right) + \frac{191490607}{46656} \zeta(6) + \frac{10358551}{43200} \zeta^2(3) \\
 & - \frac{40136}{27} a_6 + \frac{26404}{27} b_6 - \frac{700706}{675} a_4 \zeta(2) - \frac{26404}{27} a_5 \ln 2 + \frac{26404}{27} \zeta(5) \ln 2 - \frac{63749}{50} \zeta(3) \zeta(2) \ln 2 \\
 & - \frac{40723}{135} \zeta(4) \ln^2 2 + \frac{13202}{81} \zeta(3) \ln^3 2 - \frac{253201}{2700} \zeta(2) \ln^4 2 + \frac{7657}{1620} \ln^6 2 + \frac{2895304273}{435456} \zeta(7) \\
 & + \frac{670276309}{193536} \zeta(4) \zeta(3) + \frac{85933}{63} a_4 \zeta(3) + \frac{7121162687}{967680} \zeta(5) \zeta(2) - \frac{142793}{18} a_5 \zeta(2) - \frac{195848}{21} a_7 \\
 & + \frac{195848}{63} b_7 - \frac{116506}{189} d_7 - \frac{4136495}{384} \zeta(6) \ln 2 - \frac{1053568}{189} a_6 \ln 2 + \frac{233012}{189} b_6 \ln 2 \\
 & + \frac{407771}{432} \zeta^2(3) \ln 2 - \frac{8937}{2} a_4 \zeta(2) \ln 2 + \frac{833683}{3024} \zeta(5) \ln^2 2 - \frac{3995099}{6048} \zeta(3) \zeta(2) \ln^2 2 \\
 & - \frac{233012}{189} a_5 \ln^2 2 + \frac{1705273}{1512} \zeta(4) \ln^3 2 + \frac{602303}{4536} \zeta(3) \ln^4 2 - \frac{1650461}{11340} \zeta(2) \ln^5 2 + \frac{52177}{15876} \ln^7 2
 \end{aligned}$$

$$a_n = \text{Li}_n(1/2), \quad b_6 = H_{0,0,0,0,1,1}(1/2), \quad b_7 = H_{0,0,0,0,1,1}(1/2), \quad d_7 = H_{0,0,0,0,1,-1,-1}(1)$$



$$I(a) = \frac{7}{12\epsilon^4} + \frac{10}{3\epsilon^3} + \frac{121}{12\epsilon^2} + \left(\frac{1541}{72} + \frac{7}{6}\zeta(3) \right) \epsilon^{-1} + X_a^{(0)} + O(\epsilon)$$

$$I(b) = \frac{5}{8\epsilon^4} + \frac{59}{16\epsilon^3} + \left(\frac{1099}{36} + 3\zeta(2) \right) \epsilon^{-2} + \left(\frac{3781}{192} + \frac{33}{2}\zeta(2) + 6\zeta(3) \right) \epsilon^{-1} + X_b^{(0)} + O(\epsilon)$$

- $X_a^{(0)}$ and $X_b^{(0)}$ cannot be fitted with HPL of 1 and 1/2
- It must contain different analytical constants
- Closing the external line of I(a) or I(b)
 - same vacuum diagram
 - $X_a^{(0)}$ and $X_b^{(0)}$ must contain the same constants

Hyperspherical representation

$D = 2$

$$\begin{array}{c} m_1 \\ \circ \\ m_2 \end{array} \text{---} p = \int \frac{k dk}{k^2 + m_1^2} \int \frac{d\Omega_k}{(p - k_1)^2 + m_2^2} = \int_0^\infty \frac{dl}{l + m_1^2} \frac{1}{R(l, p^2, -m_2^2)}$$

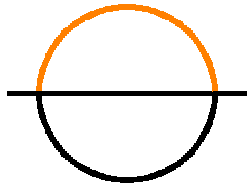
$R(x, y, z) = \sqrt{(x - y - z)^2 - 4yz}$ Källen function

$$\begin{array}{c} m_1 \\ \circ \\ m_2 \\ \circ \\ m_3 \end{array} \text{---} p = \int_0^\infty \frac{dl}{l + m_1^2} \int_0^\infty \frac{dr}{R(l, r, -m_2^2) R(r, p^2, -m_3^2)}$$

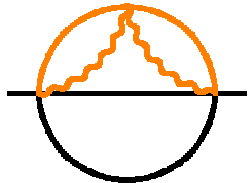
disclaimer: in general analytic continuation to timelike moments may need a deformation of the contours of the radial integrals; not needed in this case

Hyperspherical representation

$$m_i = 1$$



$$= \int_0^\infty dl \frac{1}{l+1} \int_0^\infty \frac{dr}{R(l, r, -1)R(r, -1, -1)}$$



$$\rightarrow \int_0^\infty l dl \left(\frac{\ln(l+1)}{l} \right)^2 \int_0^\infty \frac{dr}{R(l, r, -1)R(r, -1, -1)}$$

Tentative basis: ad-hoc family of integrals

$$I(n_1, n_2) = \int_0^\infty \frac{dl}{l} \ln^{n_1} l \ln^{n_2} (l+1) \dots \int_0^\infty \frac{dr}{R(l, r, -1)R(r, -1, -1)}$$

PSLQ gives

$$X_a^{(0)} = \frac{42155}{432} - \frac{380}{3}\zeta(2) + \frac{14}{3}\zeta(3) + \frac{3}{2}\zeta(4) - \frac{3}{2}I(0, 2)$$

integrating analytically $I(0,2)$ one finds (harmonic) polylogarithms of argument $e^{i\pi/3}$

$$X_a^{(0)} = \frac{42155}{432} - \frac{380}{3}\zeta(2) + \frac{14}{3}\zeta(3) + \frac{3}{2}\zeta(4) + \sqrt{3} \left(6\text{Cl}_4\left(\frac{\pi}{3}\right) - 10\zeta(2)\text{Cl}_2\left(\frac{\pi}{3}\right) \right)$$

using this basis

$$X_b^{(0)} = \frac{25033}{1152} - \frac{47}{4}\zeta(2) + \frac{69}{2}\zeta(3) + \frac{411}{8}\zeta(4) - \sqrt{3} \left(9\text{Cl}_4\left(\frac{\pi}{3}\right) + 9\zeta(2)\text{Cl}_2\left(\frac{\pi}{3}\right) \right)$$

Coefficients of $I(a)$ of power ≥ 0 have 6, 18, 61 and 199 terms, respectively.

Harmonic polylogarithms of 1, 1/2 and $e^{i\pi/3}$, $e^{2i\pi/3}$

weight	#re+im	constants
0	1 + 0	1
1	2 + 1	$\ln 2, \ln 3, \pi\sqrt{3}$
2	5 + 3	$\zeta(2), \ln^2 2, \ln 2 \ln 3, \ln^2 3, \operatorname{Re}(H_{1,-1}(e^{i\pi/3})), \pi \ln 3\sqrt{3}, \operatorname{Cl}_2\left(\frac{\pi}{3}\right)\sqrt{3}, \dots$
3	12 + 9	$\zeta(3), \operatorname{Re}(H_{1,1,-1}(e^{i\pi/3})), \operatorname{Im}(H_{0,1,1}(e^{2i\pi/3}))\sqrt{3}, \dots$
4	30 + 25	$\zeta(4), \operatorname{Re}(H_{0,0,1,1}(e^{2i\pi/3})), \operatorname{Im}(H_{0,1,1,1}(e^{2i\pi/3}))\sqrt{3}, \operatorname{Im}(H_{0,1,1,-1}(e^{i\pi/3}))\sqrt{3}, \dots$
5	76 + 68	$\zeta(5), \operatorname{Re}(H_{0,0,1,1,1}(e^{2i\pi/3})), \operatorname{Im}(H_{0,0,0,1,1}(e^{i\pi/3}))\sqrt{3}, \dots$
6	195 + 82	$\zeta(6), \operatorname{Re}(H_{0,0,1,1,0,1}(e^{i\pi/3})), \operatorname{Im}(H_{0,0,0,1,1,1}(e^{2i\pi/3}))\sqrt{3}, \dots$
7	504 + 483	$\zeta(7), \operatorname{Re}(H_{0,0,0,0,1,1,1}(e^{2i\pi/3})), \dots$
n	F_{2n+2}	

$F_{2n+2} \sim 2.618^n$ grows rapidly,
for large n , a challenge!

$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U \quad 22/671 \text{ terms}$$

$$\begin{aligned}
 V_a = & -\frac{14101}{480}\text{Cl}_4\left(\frac{\pi}{3}\right) - \frac{169703}{1440}\zeta(2)\text{Cl}_2\left(\frac{\pi}{3}\right) && \text{terms of weight 5 cancel out} \\
 & + \frac{494}{27}\text{Im}H_{0,0,0,1,-1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{494}{27}\text{Im}H_{0,0,0,1,-1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{494}{27}\text{Im}H_{0,0,0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) \\
 & + 19\text{Im}H_{0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{437}{12}\text{Im}H_{0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{29812}{297}\text{Cl}_6\left(\frac{\pi}{3}\right) \\
 & + \frac{4940}{81}a_4\text{Cl}_2\left(\frac{\pi}{3}\right) - \frac{520847}{69984}\zeta(5)\pi - \frac{129251}{81}\zeta(4)\text{Cl}_2\left(\frac{\pi}{3}\right) \\
 & - \frac{892}{15}\text{Im}H_{0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) - \frac{1784}{45}\text{Im}H_{0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) + \frac{1729}{54}\zeta(3)\text{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) \\
 & + \frac{1729}{36}\zeta(3)\text{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{837190}{729}\text{Cl}_4\left(\frac{\pi}{3}\right)\zeta(2) + \frac{25937}{4860}\zeta(3)\zeta(2)\pi \\
 & - \frac{223}{243}\zeta(4)\pi \ln 2 + \frac{892}{9}\text{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) \ln 2 + \frac{446}{3}\text{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) \ln 2 \\
 & - \frac{7925}{81}\text{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) \ln^2 2 + \frac{1235}{486}\text{Cl}_2\left(\frac{\pi}{3}\right) \ln^4 2
 \end{aligned}$$

$$\text{Cl}_n(\theta) = \text{ImLi}_n(e^{i\theta})$$

analytical fit part 2...

$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U \quad 17/825 \text{ terms}$$

$$\begin{aligned} V_b = & \frac{13487}{60} \operatorname{Re}H_{0,0,0,1,0,1} \left(e^{i\frac{\pi}{3}} \right) + \frac{13487}{60} \operatorname{Cl}_4 \left(\frac{\pi}{3} \right) \operatorname{Cl}_2 \left(\frac{\pi}{3} \right) + \frac{136781}{360} \operatorname{Cl}_2^2 \left(\frac{\pi}{3} \right) \zeta(2) \\ & + \frac{651}{4} \operatorname{Re}H_{0,0,0,1,0,1,-1} \left(e^{i\frac{\pi}{3}} \right) + 651 \operatorname{Re}H_{0,0,0,0,1,1,-1} \left(e^{i\frac{\pi}{3}} \right) - \frac{17577}{32} \operatorname{Re}H_{0,0,1,0,0,1,1} \left(e^{i\frac{2\pi}{3}} \right) \\ & - \frac{87885}{64} \operatorname{Re}H_{0,0,0,1,0,1,1} \left(e^{i\frac{2\pi}{3}} \right) - \frac{17577}{8} \operatorname{Re}H_{0,0,0,0,1,1,1} \left(e^{i\frac{2\pi}{3}} \right) \\ & + \frac{651}{4} \operatorname{Cl}_4 \left(\frac{\pi}{3} \right) \operatorname{Im}H_{0,1,-1} \left(e^{i\frac{\pi}{3}} \right) + \frac{1953}{8} \operatorname{Cl}_4 \left(\frac{\pi}{3} \right) \operatorname{Im}H_{0,1,1} \left(e^{i\frac{2\pi}{3}} \right) + \frac{31465}{176} \operatorname{Cl}_6 \left(\frac{\pi}{3} \right) \pi \\ & + \frac{211}{4} \operatorname{Re}H_{0,1,0,1,-1} \left(e^{i\frac{\pi}{3}} \right) \zeta(2) + \frac{211}{2} \operatorname{Re}H_{0,0,1,1,-1} \left(e^{i\frac{\pi}{3}} \right) \zeta(2) \\ & + \frac{1899}{16} \operatorname{Re}H_{0,1,0,1,1} \left(e^{i\frac{2\pi}{3}} \right) \zeta(2) + \frac{1899}{8} \operatorname{Re}H_{0,0,1,1,1} \left(e^{i\frac{2\pi}{3}} \right) \zeta(2) \\ & + \frac{211}{4} \operatorname{Im}H_{0,1,-1} \left(e^{i\frac{\pi}{3}} \right) \operatorname{Cl}_2 \left(\frac{\pi}{3} \right) \zeta(2) + \frac{633}{8} \operatorname{Im}H_{0,1,1} \left(e^{i\frac{2\pi}{3}} \right) \operatorname{Cl}_2 \left(\frac{\pi}{3} \right) \zeta(2) \end{aligned}$$

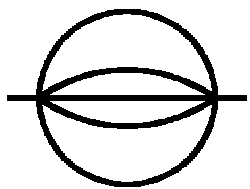
weight	#re+im	constants
0	1 + 0	1
1	1 + 1	$\ln 2, \pi$
2	2 + 2	$\zeta(2), \ln^2 2, \pi \ln 2, \beta_2$
3	4 + 4	$\zeta(3), \pi\beta_2, \text{Im}(H_{0,1,1}(e^{i\pi/2})), \dots$
4	8 + 8	$\zeta(4), \beta_2^2, \beta_4, \text{Im}(H_{0,1,1,1}(e^{i\pi/2}))\sqrt{3}, \dots$
5	16 + 16	$\zeta(5), \text{Re}(H_{0,1,0,1,1}(e^{i\pi/2})), \text{Im}(H_{0,0,0,1,1}(e^{i\pi/2})), \dots$
6	32 + 32	$\zeta(6), \text{Re}(H_{0,0,0,1,0,1}(e^{i\pi/2})), \text{Im}(H_{0,0,0,1,1,1}(e^{i\pi/2})), \dots$
7	64 + 64	$\zeta(7), \text{Re}(H_{0,0,0,1,0,1,1}(e^{i\pi/2})), \text{Im}(H_{0,0,0,0,0,1,1}(e^{i\pi/2})), \dots$
n	2^n	

$\beta_2 = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \dots$ is the Catalan's constant

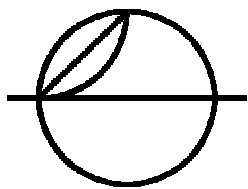
$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U \quad 5/256 \text{ terms}$$

$$\begin{aligned} W_b = & -\frac{28276}{25}\zeta(2)\text{Cl}_2\left(\frac{\pi}{2}\right)^2 \\ & +104\left(4\text{Re}H_{0,1,0,1,1}\left(e^{i\frac{\pi}{2}}\right)\zeta(2) + 4\text{Im}H_{0,1,1}\left(e^{i\frac{\pi}{2}}\right)\text{Cl}_2\left(\frac{\pi}{2}\right)\zeta(2) \right. \\ & \left. -2\text{Cl}_4\left(\frac{\pi}{2}\right)\zeta(2)\pi + \text{Cl}_2^2\left(\frac{\pi}{2}\right)\zeta(2)\ln 2\right) \end{aligned}$$

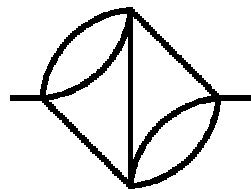
Elliptic master integrals



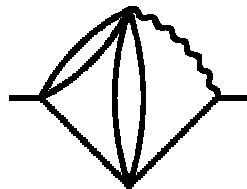
(a)



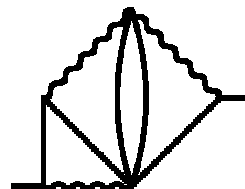
(b)



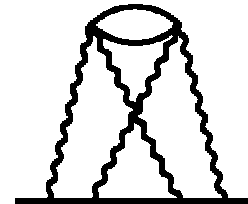
(c)



(d)



(e)

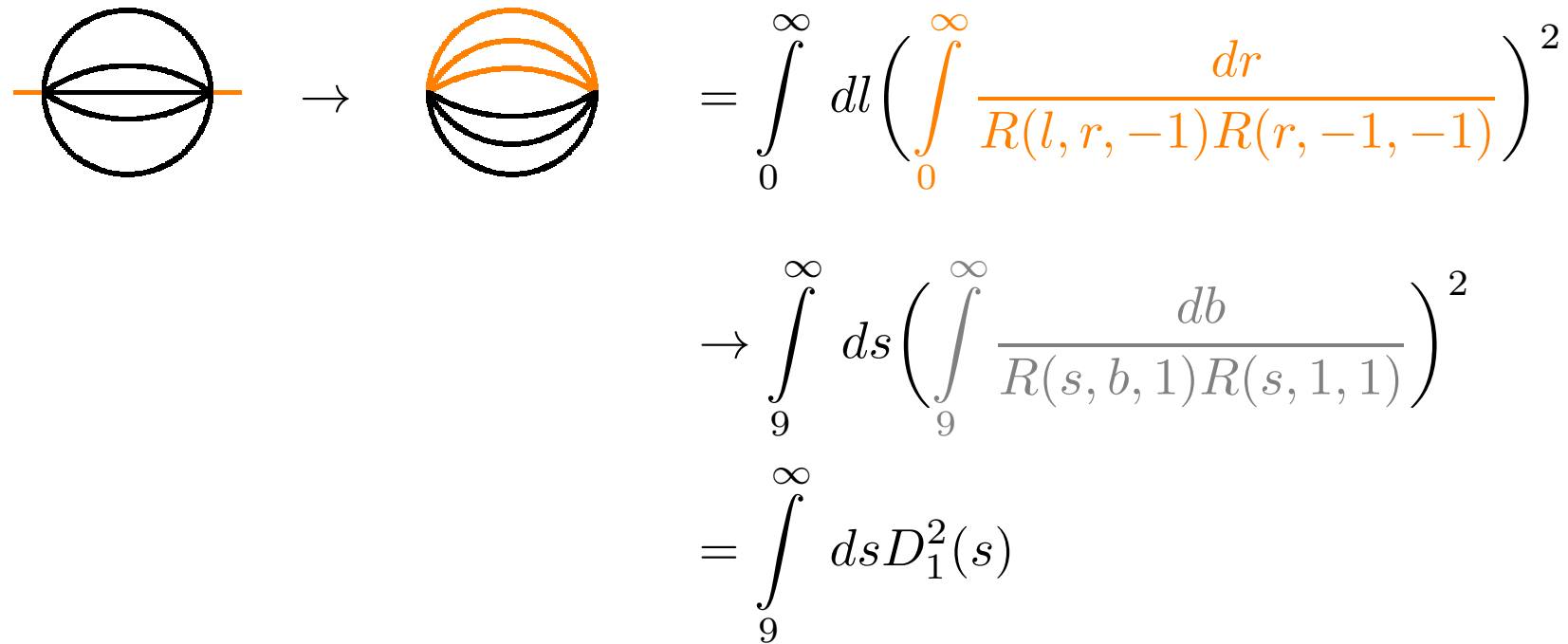


(f)



(g)

Elliptical: closing



$$\begin{aligned}
 &= \int_0^{\infty} dl \left(\int_0^{\infty} \frac{dr}{R(l, r, -1)R(r, -1, -1)} \right)^2 \\
 &\rightarrow \int_9^{\infty} ds \left(\int_9^{\infty} \frac{db}{R(s, b, 1)R(s, 1, 1)} \right)^2 \\
 &= \int_9^{\infty} ds D_1^2(s)
 \end{aligned}$$

$$D_1(s) = \frac{2}{\sqrt{(\sqrt{s} + 3)(\sqrt{s} - 1)^3}} K \left(\frac{(\sqrt{s} - 3)(\sqrt{s} + 1)^3}{(\sqrt{s} + 3)(\sqrt{s} - 1)^3} \right),$$

$$D_2(s) = \frac{2}{\sqrt{(\sqrt{s} + 3)(\sqrt{s} - 1)^3}} K \left(1 - \frac{(\sqrt{s} - 3)(\sqrt{s} + 1)^3}{(\sqrt{s} + 3)(\sqrt{s} - 1)^3} \right);$$

$K(x)$ is the complete elliptic integral of the first kind; $D_1(s) \sim$ discontinuity of the 2-loop sunrise diagram with equal masses in $D = 2$ dimensions.

$$f_1(i, j, k) = \int_1^9 ds D_1^2(s) \left(s - \frac{9}{5} \right) \ln^i(9-s) \ln^j(s-1) \ln^k(s) ,$$

$$f_2(i, j, k) = \int_1^9 ds D_1(s) \operatorname{Re} \left(\sqrt{3} D_2(s) \right) \left(s - \frac{9}{5} \right) \ln^i(9-s) \ln^j(s-1) \ln^k(s) ,$$

$(s - 9/5)$ ad-hoc factor

constant term of $I(a)$ fitted with integrals of elliptic integrals squared (SL,2008).

$$I(a, D=4-2\epsilon) = -\frac{5}{2\epsilon^4} - \frac{45}{4\epsilon^3} - \frac{4255}{144\epsilon^2} - \frac{106147}{1728\epsilon} + \frac{\pi\sqrt{3}}{240} (297B_3 - 1477C_3) - \frac{2320981}{20736} + O(\epsilon)$$

$$I(a, D=2-2\epsilon) = \sqrt{3}\pi B_3 + M_{501}\epsilon + M_{602}\epsilon^2 - M_{702}\epsilon^3 + \dots$$

$$A_3 = \int_0^1 dx \frac{K_c(x)K_c(1-x)}{\sqrt{1-x}} = \frac{2\pi^{\frac{3}{2}}}{3} \left(\frac{\Gamma^2(\frac{7}{6})\Gamma(\frac{1}{3})}{\Gamma^2(\frac{2}{3})\Gamma(\frac{5}{6})} {}_4F_3 \left(\begin{matrix} \frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{2} \\ \frac{5}{6}, \frac{5}{6}, \frac{2}{3} \end{matrix}; 1 \right) - \frac{\Gamma^2(\frac{5}{6})\Gamma(-\frac{1}{3})}{\Gamma^2(\frac{1}{3})\Gamma(\frac{1}{6})} {}_4F_3 \left(\begin{matrix} \frac{1}{2}, \frac{2}{3}, \frac{2}{3}, \frac{5}{6} \\ \frac{7}{6}, \frac{7}{6}, \frac{4}{3} \end{matrix}; 1 \right) \right)$$

$$B_3 = \int_0^1 dx \frac{K_c^2(x)}{\sqrt{1-x}} = \frac{4\pi^{\frac{3}{2}}}{3} \left(\frac{\Gamma^2(\frac{7}{6})\Gamma(\frac{1}{3})}{\Gamma^2(\frac{2}{3})\Gamma(\frac{5}{6})} {}_4F_3 \left(\begin{matrix} \frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{2} \\ \frac{5}{6}, \frac{5}{6}, \frac{2}{3} \end{matrix}; 1 \right) + \frac{\Gamma^2(\frac{5}{6})\Gamma(-\frac{1}{3})}{\Gamma^2(\frac{1}{3})\Gamma(\frac{1}{6})} {}_4F_3 \left(\begin{matrix} \frac{1}{2}, \frac{2}{3}, \frac{2}{3}, \frac{5}{6} \\ \frac{7}{6}, \frac{7}{6}, \frac{4}{3} \end{matrix}; 1 \right) \right)$$

$$C_3 = \int_0^1 dx \frac{E_c^2(x)}{\sqrt{1-x}} = \frac{486\pi^2}{1925} {}_7F_6 \left(\begin{matrix} \frac{7}{4}, -\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{3}{2}, \frac{3}{2} \\ \frac{3}{4}, 1, \frac{7}{6}, \frac{11}{6}, \frac{13}{6}, \frac{17}{6} \end{matrix}; 1 \right),$$

$$K_c(x) = \frac{2\pi}{\sqrt{27}} {}_2F_1 \left(\begin{matrix} \frac{1}{3}, \frac{2}{3} \\ 1 \end{matrix}; x \right), \quad E_c(x) = \frac{2\pi}{\sqrt{27}} {}_2F_1 \left(\begin{matrix} \frac{1}{3}, -\frac{1}{3} \\ 1 \end{matrix}; x \right).$$

A_3 cancels out in the diagram contributions

A_3 and B_3 seem to be both irreducible

$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U$$

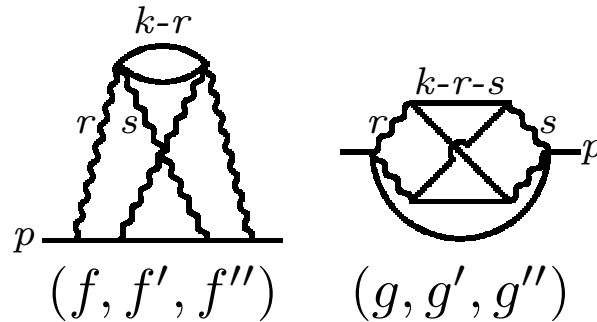
$$\begin{aligned} E_a = & \pi \left(-\frac{28458503}{691200} B_3 + \frac{250077961}{18662400} C_3 \right) + \frac{483913}{77760} \pi f_2(0, 0, 1) \\ & + \pi \left(\frac{4715}{1944} \ln 2 f_2(0, 0, 1) + \frac{270433}{10935} f_2(0, 2, 0) - \frac{188147}{4860} f_2(0, 1, 1) + \frac{188147}{12960} f_2(0, 0, 2) \right) \\ & + \pi \left(\frac{826595}{248832} \zeta(2) f_2(0, 0, 1) - \frac{5525}{432} \ln 2 f_2(0, 0, 2) + \frac{5525}{162} \ln 2 f_2(0, 1, 1) \right. \\ & - \frac{5525}{243} \ln 2 f_2(0, 2, 0) + \frac{526015}{248832} f_2(0, 0, 3) - \frac{4675}{768} f_2(0, 1, 2) + \frac{1805965}{248832} f_2(0, 2, 1) \\ & - \frac{3710675}{1119744} f_2(0, 3, 0) - \frac{75145}{124416} f_2(1, 0, 2) - \frac{213635}{124416} f_2(1, 1, 1) + \frac{168455}{62208} f_2(1, 2, 0) \\ & \left. + \frac{69245}{124416} f_2(2, 1, 0) \right) \end{aligned}$$

$$\begin{aligned} E_b = & -\frac{4715}{1458} \zeta(2) f_1(0, 0, 1) \\ & + \zeta(2) \left(\frac{2541575}{82944} f_1(0, 0, 2) - \frac{556445}{6912} f_1(0, 1, 1) + \frac{54515}{972} f_1(0, 2, 0) - \frac{75145}{20736} f_1(1, 0, 1) \right) . \end{aligned}$$

$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U$$

The term containing the ϵ^0 coefficients of the ϵ -expansion of six master integrals (see f, f', f'', g, g', g''):

$$U = -\frac{541}{300}C_{81a} - \frac{629}{60}C_{81b} + \frac{49}{3}C_{81c} - \frac{327}{160}C_{83a} + \frac{49}{36}C_{83b} + \frac{37}{6}C_{83c} .$$



(f, f', f'') and (g, g', g'') have numerators respectively equal to $(1, p.k, (p.k)^2)$

These master integrals appear in topologies 81 and 83 (gauge-invariant sets 24 and 25, vacuum polarization diagrams containing a light-light scattering).

Conclusions

- fitting analytically families of master integrals is complicated
- it needs very high precision numerical values
- it needs a deep analysis of some key master integrals
- in the case of $g=2$ only a few percent of the total number of elements of the basis with polylogarithms of complex argument are needed
- some guide to promptly identify only the needed elements of the basis would be useful

Conclusions

The End