Universität Zürich[UZH]

MITP *μe workshop*, 19th-23th February 2018

Two-loop master integrals for μe-scattering in QED

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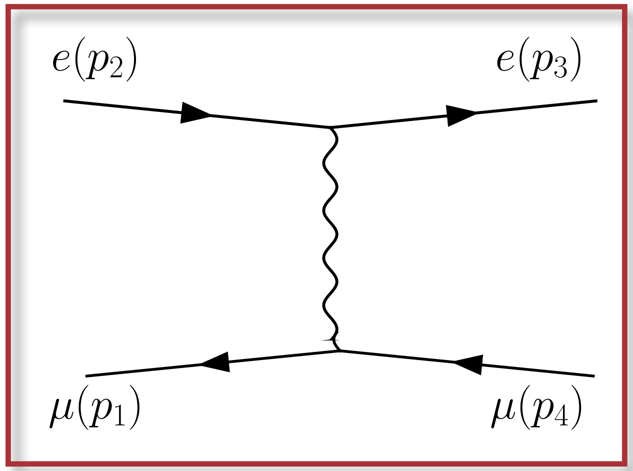
Based on: P. Mastrolia, M. Passera, AP and U.Schubert,10.1007/JHEP11(2017)198
In collaboration with G.Ossola, P. Mastrolia, M. Passera, J.Ronca, U.Schubert, W.J. Torres Bobadilla

μe-scattering

$$\mu^+(p_1) + e^-(p_2) \to \mu^+(p_4) + e^-(p_3)$$

$$\boxed{\begin{gathered} s = (p_1+p_2)^2, \quad t = (p_2-p_3)^2, \quad u = -s - t + 2m^2 - 2m_e^2 \\ \frac{d\sigma_{\mathrm{LO}}}{dt} = 4\pi\alpha^2 \frac{\left(m^2+m_e^2\right)^2 - su + t^2/2}{t^2\lambda\left(s, m^2, m_e^2\right)} \end{gathered}}$$



$$s > (m_e + m)^2$$

$$-\lambda(s, m^2, m_e^2)/s < t < 0$$

- MUonE experiment demands for more accurate predictions:

$$\sigma = \sigma_{\mathrm{LO}} + \sigma_{\mathrm{NLO}} + \sigma_{\mathrm{NNLO}} + \ldots$$

$$\sigma_{\mathrm{NLO}} = \int \mathrm{d}\phi_2 \underbrace{2Re\ \mathcal{M}^{(0)\,*}\mathcal{M}^{(1)}}_{\text{Virtual}} + \int \underbrace{\mathrm{d}\phi_3 |\mathcal{M}_\gamma^{(0)}|^2}_{\text{Real}}$$

Nikishov (61), Eriksson (61), Eriksson, Larsson, Rinander (63), von Nieuwenhuizen (71)

see Carloni Calame's talk

$$\sigma_{\mathrm{NNLO}} = \int \mathrm{d}\phi_2 \left(2Re\ \underbrace{\mathcal{M}^{(0)\,*}\mathcal{M}^{(2)} + |\mathcal{M}^{(1)}|^2}_{\text{Virtual}}\right) + \int \mathrm{d}\phi_3 \underbrace{Re\ \mathcal{M}_\gamma^{(0)\,*}\mathcal{M}_\gamma^{(1)}}_{\text{Real-Virtual}} + \int \underbrace{\mathrm{d}\phi_4 |\mathcal{M}_{\gamma\gamma}^{(0)}|^2}_{\text{Double-Real}}$$

see Torres' and Greiner's talks

- **This talk**: evaluation of loop integrals needed for $\mathcal{M}^{(2)}$

Outline

- Loop amplitudes computation

- Integration-by-parts and differential equations for loop integrals

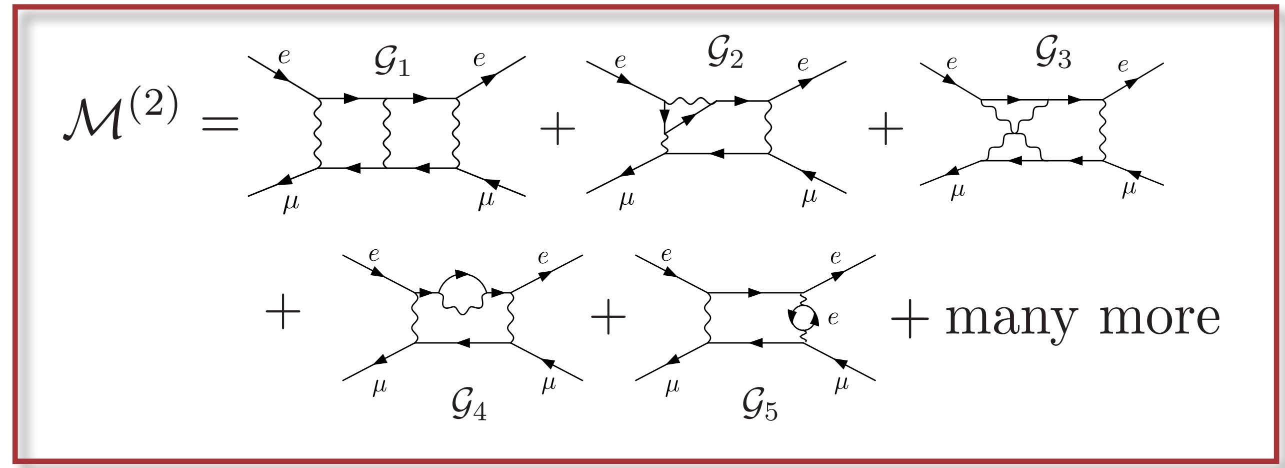
- Results for NNLO μe-scattering

- Conclusions and outlook

Multiloop amplitudes

- **Diagrammatic** approach to scattering amplitudes in perturbation theory:

$$\mathcal{M}^{(\ell)}(p_1, \ldots, p_N) = \sum_{i=1}^{n_{\text{graph}}} \mathcal{G}_i(p_1, \ldots, p_N)$$



- Amplitude expressed in terms of scalar **Feynman integrals**:

$$\mathcal{M}^{(\ell)}(s_{ij}, \epsilon) = \sum_k a_k(s_{ij}, \epsilon) \mathcal{I}_k(s_{ij}, \epsilon)$$

$$\boxed{\mathcal{I}_k(s_{ij}, \epsilon) = \int \prod_{i=1}^{\ell} \mathrm{d}^d q_i \frac{1}{D_1^{a_1} \cdots D_{n_k}^{a_{n_k}}}\,, \qquad D_i = (l_i^2 - m_i^2)\,, \qquad a_i \in \mathbb{Z}}$$

- Different techniques to extract $a_k(s_{ij}, \epsilon)$:
    - Integral decomposition (projection into **form factors**)
    - **Integrand** decomposition Ossola, Papadopoulus, Pittau (07) Ellis,Giele Kunszt (08), Mastrolia, Ossola (11), Zhang (12) Mastrolia, Mirabella, Ossola, Peraro (12) Mastrolia, Peraro, AP (16), …
    - **Generalised** and **numerical unitarity** Bern, Dixon, Dunbar, Kosower (94), Cachazo, Svrcek, Witten (04), Britto, Cachzao, Feng (05), … Ita (15), Abreu, Febres Cordero, Ita, Jaquier, Page (17), …

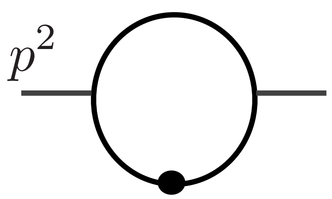
Integration-by-parts

- Feynman integrals in DR obey **Integration-by-Parts** identities:

$$\int \prod_{i=1}^{\ell} \mathrm{d}^d q_i \frac{\partial}{\partial q_j^{\mu}} \left(\frac{v_k^{\mu}}{D_1^{a_1} \cdots D_{n_k}^{a_{n_k}}} \right) = 0 \,, \qquad v_k^{\mu} \in \{q_k^{\mu}, p_k^{\mu}\}$$

Chetyrkin, Tkachov (81)

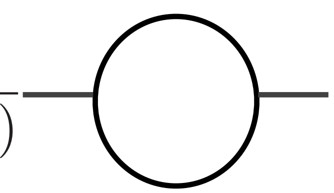
- IBPs generate identities between different Feynman integrals $\mathcal{I}_k(s_{ij}, \epsilon)$



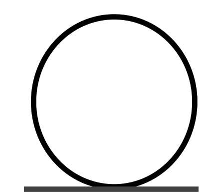
$$= \int \frac{\mathrm{d}^d q}{D_1^2 D_2} = \int \frac{\mathrm{d}^d q}{(q^2 - m^2)^2[(q-p)^2 - m^2]}$$

$$= \frac{d-3}{(p^2 - 4m^2)} \int \frac{\mathrm{d}^d q}{D_1 D_2} - \frac{d-2}{2m^2(p^2 - 4m^2)} \int \frac{\mathrm{d}^d q}{D_1}$$

$$= \frac{d-3}{(p^2 - 4m^2)}$$



$$- \frac{d-2}{2m^2(p^2 - 4m^2)}$$



- IBPs produce huge overdetermined systems. Solved through the **Laporta algorithm**

- Reduction automatised in several public codes: **AIR** (Anastasiou, Lazopoulos 04), **FIRE** (Smirnov 08), **Reduze** (Studerus 10, + von Manteuffel 12), **LiteRed** (Lee 12), **Kira** (Maierhoefer, Usovitsch, Uber 17)

Integration-by-parts

- All Feynman integrals expressed in terms of a finite number of **master integrals**

$$\mathcal{I}_k(s_{ij},\epsilon) = \sum_{n=1}^{N} b_n(s_{ij},\epsilon) f_k(s_{ij},\epsilon)$$

- Loop amplitude reduced to a linear combination of the master integrals:

$$\mathcal{M}^{(\ell)}(p_1,\ldots,p_N) = \sum_{i=1}^{n_{\text{graph}}} \mathcal{G}_i(p_1,\ldots,p_N)$$

amplitude projection

$$\mathcal{M}^{(\ell)}(s_{ij},\epsilon) = \sum_{k} a_k(s_{ij},\epsilon)\mathcal{I}_k(s_{ij},\epsilon)$$

IBPs

$$\mathcal{M}^{(\ell)}(s_{ij},\epsilon) = \sum_{n=1}^{N} c_n(s_{ij},\epsilon) f_k(s_{ij},\epsilon)$$

- Typical two-loop case: **hundreds** of diagrams reduced to **dozens** of master integrals

- **Analytic evaluation** of the master integrals:

  - direct integration (Feynman parameters, Mellin-Barnes) Smirnov (99), Tausk (99), Czakon (06), A.V. Smirnov, V.A. Smirnov (09)
  - indirect techniques (difference equations, differential equations)

Laporta (00)-(01),
Lee A.V. Smirnov, V.A. Smirnov (10)

Differential equations method

- **Master integrals** $f_k(s_{ij},\epsilon)$ form a basis of the vector space of Feynman integrals

$$\boxed{\mathcal{I}_k(s_{ij},\epsilon) = \sum_{n=1}^{N} b_n(s_{ij},\epsilon) f_k(s_{ij},\epsilon)}$$

- Master integrals fulfil 1st order coupled **differential equations** in the kinematic invariants:

  - Differentiate $f_k(s_{ij},\epsilon)$ w.r.t. s_{ij}
  - $\partial f_k(s_{ij},\epsilon)$ belongs to the space spanned by $\vec{f} = (f_1, f_2, \ldots, f_N)$
  - Use IBPs to express $\partial f_k(s_{ij},\epsilon)$ in terms of the master integrals

$$\boxed{\frac{\partial}{\partial s_{ij}} \vec{f}(s_{mn},\epsilon) = \mathbb{A}_{ij}(s_{mn},\epsilon) \vec{f}(s_{mn},\epsilon)}$$

Kotikov (91) Remiddi (97),
Gehrmann, Remiddi (00)

- $\mathbb{A}_{ij}(s_{mn},\epsilon)$ is **block-triangular** and has **rational** dependence on s_{mn} and ϵ

- The system can be solved **bottom-up**: previously determined integrals enter the inhomogeneous part of the DEQ

Differential equations method

- Determination of the solution with **exact dependence** on ϵ generally not possibile

- **But** we are interested in the **Laurent expansion** of the integrals around $\epsilon \sim 0$:

$$\vec{f}(\vec{x},\epsilon) = \sum_{k=-n_{\mathrm{l.p.}}}^{\infty} \vec{f}^{(k)}(\vec{x})\epsilon^k \qquad\qquad \vec{x} = \{s_{ij}/m^2\}$$

- Expand the DEQs and obtain chained equations for the coefficients $\vec{f}^{(k)}(\vec{x})$

- **Conjecture**: suppose we have a DEQ of the form

$$\boxed{\mathrm{d}\vec{f}(\vec{x},\epsilon) = \epsilon \mathrm{d}\,\mathbb{A}(\vec{x})\vec{f}(\vec{x},\epsilon) \quad \Rightarrow \quad \partial_i \vec{f}(\vec{x},\epsilon) = \epsilon\, \partial_i \mathbb{A}(\vec{x})\vec{f}(\vec{x},\epsilon)}$$

Henn (13)

- The DEQs for $\vec{f}^{(k)}(\vec{x})$ **decouple** order-by-order in ϵ:

$$\mathrm{d}\vec{f}^{(k)}(\vec{x}) = \mathrm{d}\,\mathbb{A}(\vec{x})\vec{f}^{(k-1)}(\vec{x})$$

-System solved by **matrix multiplication** $(n_{\mathrm{l.p.}} = 0)$

$$\boxed{\vec{f}(\vec{x},\epsilon) = \left(1 + \sum_{k=1}^{\infty} P_k(\vec{x})\epsilon^k\right)\vec{f}(\vec{x}_0,\epsilon) \qquad P_k(\vec{x}) = \int_\gamma \underbrace{\mathrm{d}\,\mathbb{A}(\vec{x})\,\mathrm{d}\,\mathbb{A}(\vec{x})\,\ldots\,\mathrm{d}\,\mathbb{A}(\vec{x})}_{\text{k times}}}$$

General solution

- In the optimal case, the kinematic matrix of the DEQs is in **dlog-form**:

$$\mathrm{d}\vec{f}(\vec{x}) = \epsilon \mathrm{d}\mathbb{A}(\vec{x})\vec{f}(\vec{x}) \qquad \mathrm{d}\mathbb{A}(\vec{x}) = \sum_{i=1}^{m} \mathbb{M}_i \mathrm{d}\log\eta_i(\vec{x})$$

Henn (13)

- Refers to such system as **canonical**: if such a basis is found, the determination of the solution is **algorithmic**

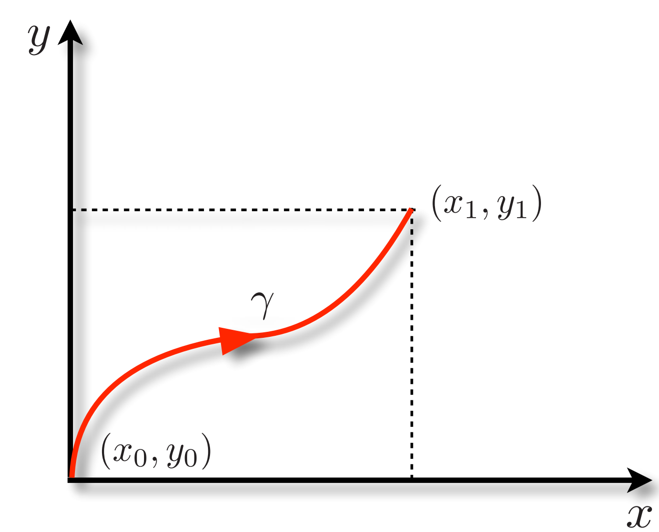
- The **letters** of the **alphabet** η_i and the coefficient matrices $\mathbb{M}_i$ encode all information on the general solution

- The entries of $P_k(\vec{x})$ are written in terms of **Chen iterated integrals**

$$\mathcal{C}^{[\gamma]}_{i_k,\ldots,i_1} = \int_{0\le t_1\le\ldots\le t_k\le 1} g^{\gamma}_{i_k}(t_k)\ldots g^{\gamma}_{i_1}(t_1)\,\mathrm{d}t_1\ldots\,\mathrm{d}t_k$$

$$g^{\gamma}_i(t) = \frac{d}{dt}\log\eta_i(\gamma(t))$$

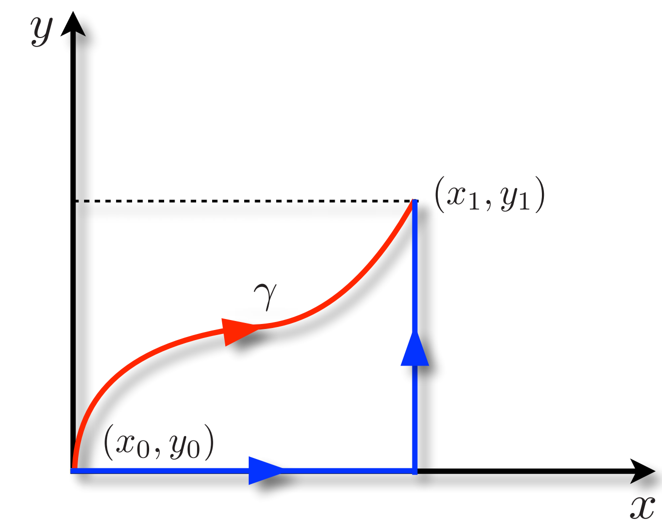
Chen (77)



Generalised polylogarithms

- If the alphabet is **rational** we can factor it and integrate on a piecewise **linear path**:

$$\eta_k(\vec{x}) = \prod_{j_k=1}^{m_k} (x_i - \omega_{j_k}) \implies \mathbb{A}_i(\vec{x}) = \sum_{j=1}^{m} \mathbb{P}_j \frac{1}{x_i - \omega_j}$$



-The solution is expressed in terms of **generalised polylogarithms**:

$$\boxed{\begin{aligned} G(\vec{\omega}_n; x) &= \int_0^x \mathrm{d}t \frac{1}{t-\omega_1} G(\vec{\omega}_{n-1}; t)\,, \qquad n > 0 \\ G(\vec{0}_n; x) &= \frac{1}{n!} \log^n x\,, \end{aligned}}$$

Goncharov (98)-(01), Remiddi Vermaseren (00), Gehrmann Remiddi (01), Vollinga Weinzierl (05)

- GPLs are well known functions in particle physics: **analyticity** properties and **algebraic** structure under complete control

- Accurate automatised numerical evaluation is available (GiNaC)

Boundary conditions

- General solution must be complemented with suitable **boundary conditions**

$$\vec{f}(\vec{x},\epsilon) = \left(1 + \sum_{k=1}^{\infty} P_k(\vec{x})\epsilon^k\right) \boxed{\vec{f}(\vec{x}_0,\epsilon)}$$

- Few integrals are know analytically in the initial PS-point: **external input**

- The presence of **unphysical singularities** in the DEQs gives **internal** information on the boundary terms

- If η_i is a pseudo-threshold :

$$\boxed{\lim_{\eta_i \to 0} \mathbb{M}_i \vec{f}(\vec{x},\epsilon) = 0}$$

- **All-order** (retarded) relation between boundary constants

  - e.g. at $\mathcal{O}(\epsilon)$: $\quad \vec{f}^{(1)}(\vec{x}) = \int_\gamma \mathrm{d}\mathbb{A}\, \vec{f}^{(0)}(\vec{x}_0) + \vec{f}^{(1)}(\vec{x}_0)$

$$\lim_{\eta_i \to 0} \vec{f}^{(1)}(\vec{x}) \sim \lim_{\eta_i \to 0} \int_\gamma \mathrm{d}\log \eta_i \mathbb{M}_i\, \vec{f}^{(0)}(\vec{x}_0) \quad \Rightarrow \quad \lim_{\eta_i \to 0} \mathbb{M}_i\, \vec{f}^{(0)}(\vec{x}_0) = 0$$

Finding the canonical form

- **Change of basis** of master integrals : $\vec{f}(\vec{x},\epsilon) = \mathbb{B}(\vec{x},\epsilon)\vec{g}(\vec{x},\epsilon)$

$$\partial_i \vec{f}(\vec{x},\epsilon) = \partial_i \mathbb{A}(\vec{x},\epsilon)\vec{f}(\vec{x},\epsilon) \quad \Rightarrow \quad \partial_i \vec{g}(\vec{x},\epsilon) = \left(\mathbb{B}^{-1}\mathbb{A}\mathbb{B} + \mathbb{B}^{-1}\partial_i\mathbb{B}\right)\vec{g}(\vec{x},\epsilon)$$

- **Ansatz**: start from DEQs **linear** in ϵ:

$$\boxed{\partial_i \vec{g}(\vec{x},\epsilon) = \left(\mathbb{A}_{0,i}(\vec{x}) + \epsilon\mathbb{A}_{1,i}(\vec{x})\right)\vec{g}(\vec{x},\epsilon)}$$

- If we find a matrix solution of the DEQs at $\epsilon = 0$:

$$\partial_i \mathbb{B}(\vec{x}) = \mathbb{A}_0(x)\mathbb{B}(\vec{x})$$

- We can **rotate** the master integrals to **canonical basis**:

$$\partial_i \vec{g}(\vec{x},\epsilon) = \left(\mathbb{B}^{-1}(\cancel{\mathbb{A}_0} + \epsilon\mathbb{A}_0)\mathbb{B} + \mathbb{B}^{-1}\cancel{\partial_i\mathbb{B}}\right)\vec{g}(\vec{x},\epsilon)$$

$$\Downarrow$$

$$\boxed{\partial_i \vec{g}(\vec{x},\epsilon) = \epsilon\hat{\mathbb{A}}_0(\vec{x})\vec{g}(\vec{x},\epsilon)}$$

Magnus exponential method

- 1st order homogeneous matrix differential equation:

Argeri, Di Vita, Mastrolia et al (14)

$$\partial_x \mathbb{B}(x) = \mathbb{A}_0(x)\mathbb{B}(x)$$

- The (approximate) solution written in terms of the **Magnus exponential**

$$\mathbb{B}(x) = e^{\Omega[\mathbb{A}_0](x)} \equiv \mathbb{1} + \Omega[\mathbb{A}_0](x) + \frac{1}{2!}\Omega[\mathbb{A}_0](x)\,\Omega[\mathbb{A}_0](x) + \ldots$$

$$\Omega[\mathbb{A}_0](x) = \sum_{n=0}^{\infty} \Omega_n[\mathbb{A}_0](x)$$

$$\Omega_1[\mathbb{A}_0](x) = \int \mathrm{d}x_1\, \mathbb{A}_0(x_1),$$

$$\Omega_2[\mathbb{A}_0](x) = \frac{1}{2}\int \mathrm{d}x_1 \int \mathrm{d}x_2\, [\mathbb{A}_0(x_1), \mathbb{A}_0(x_2)],$$

$$\Omega_3[\mathbb{A}_0](x) = \frac{1}{6}\int \mathrm{d}x_1 \int \mathrm{d}x_2 \int \mathrm{d}x_3\, [\mathbb{A}_0(x_1), [\mathbb{A}_0(x_2), \mathbb{A}_0(x_3)]] + [\mathbb{A}_0(x_3), [\mathbb{A}_0(x_2), \mathbb{A}_0(x_1)]]$$

$$\ldots$$

- If $\Omega_n[\mathbb{A}_0](x) = 0$ for $n > n_{\max}$, Magnus exponential **exact solution** of the DEQs at $\epsilon = 0$
- Use it to rotate the DEQs to **canonical form**:

$$\partial_x \vec{f}(x,\epsilon) = [\mathbb{A}_0 + \epsilon\mathbb{A}_1]\vec{f}(x,\epsilon) \qquad \vec{f}(x,\epsilon) = e^{\Omega[\mathbb{A}_0](x)}\vec{g}(x,\epsilon)$$

$$\boxed{\vec{g}(x,\epsilon) = \epsilon\, e^{\Omega[\mathbb{A}_0]}\mathbb{A}_1 e^{-\Omega[\mathbb{A}_0]}\vec{g}(x,\epsilon)}$$

Canonical systems

- Magnus exponential applied to several multiloop multiscale problems:
    - Two loop QED vertex **Argeri, Di Vita, Mastrolia et al (14)**
    - Three-loop three jet production, H+jet (EFT) **Di Vita, Mastrolia, Schubert et al (14)**
    - Two-loop mixed EW-QCD corrections to Drell-Yan **Bonciani, Di Vita, Mastrolia, et al (16)**
    - Two loop mixed EW-QCD to WWH, WWZ(γ^*) **Di Vita, Mastrolia, AP et al (17)**
    - Two loop μe-scattering in QED **Mastrolia, Passera, AP et al (17)**

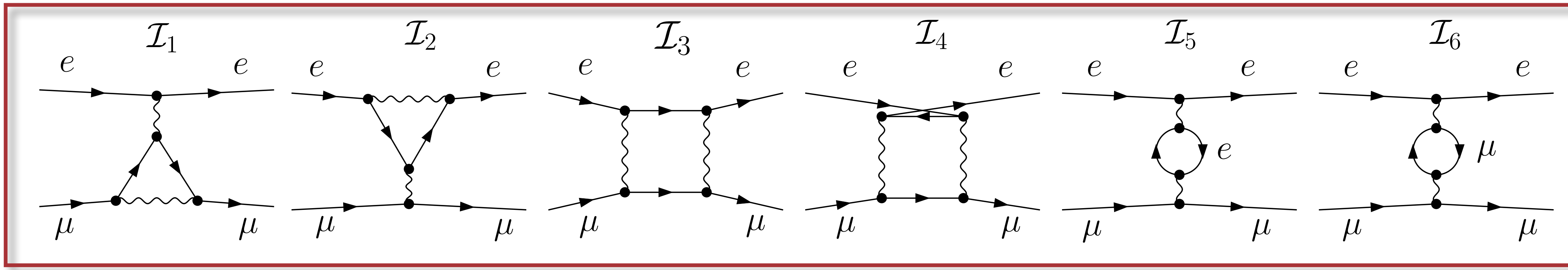
- Several other strategies for finding canonical forms are available:
    - Basis with **unit leading singularity** **Henn (13)**
    - **Bottom-up** construction of rational change of basis **Gehrmann, von Manteuffel, Tancredi et at (14)**
    - **Reduction** to fuchsian form and eigenvalue normalisation **Lee (15), Lee and Smirnov (16)**
    - Factorisation of the **Picard-Fuchs** operator **Adams, Chaubey, Weinzierl (17)**

- Publicly available codes: **Canonica**, **Fuchsia**, **Epsilon**

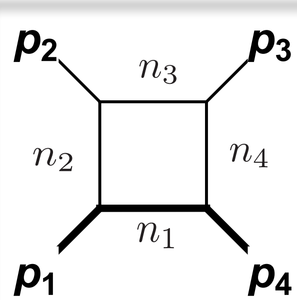
- Extension of ϵ-factorised basis beyond multiple polylogarithms
    - **Maximal Unitarity cuts** of Feynman integrals **Bonciani, Del Duca, Frellesvig et al (16), AP and Tancredi (16), A.P. and Tancredi (17)**
    - **Transcendental** change of basis **Adams, Weinzierl (18)**

μe: one-loop virtual corrections

- **NLO virtual** contributions:



- Define one-loop **integral family** (includes $\mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_5$) :



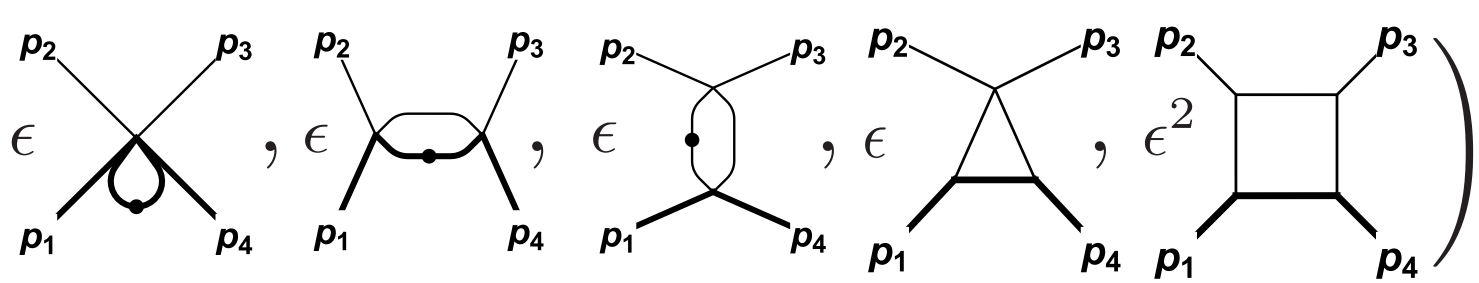
$$= \int \widetilde{\mathrm{d}^d k_1} \, \frac{1}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4}}$$

$$D_1 = k_1^2 - m^2$$
$$D_2 = (k_1 + p_1)^2$$
$$D_3 = (k_1 + p_1 + p_2)^2$$
$$D_4 = (k_1 + p_4)^2$$

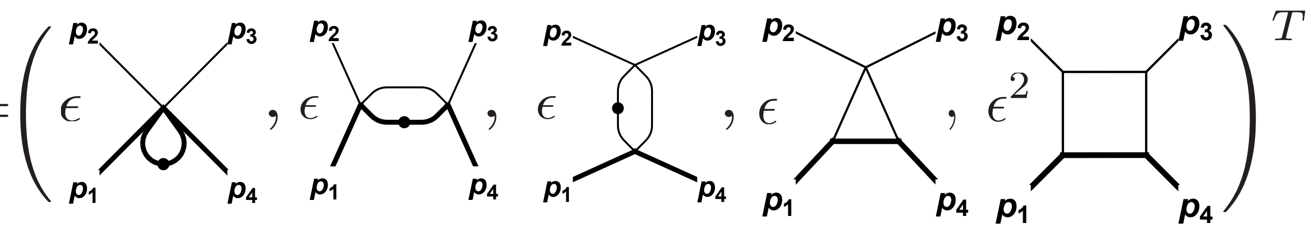
$$\boxed{m_e \sim 0}$$

- Use IBPs to determine a **basis** of master integrals:

$$\vec{f}(s, t, m^2, \epsilon) = \left(\epsilon \, \cdot, \ \epsilon \, \cdot, \ \epsilon \, \cdot, \ \epsilon \, \cdot, \ \epsilon^2 \, \cdot \right)^T$$



Differential equations

$\vec{f}(s,t,m^2,\epsilon) =$ 

- Combine kinematical variables in two dimensionless parameters:

$$\boxed{-\frac{s}{m^2} = x\,, \qquad -\frac{t}{m^2} = \frac{(1-y)^2}{y}}$$

- Derive systems of DEQs in x and y:

$$\partial_x \vec{f}(x,y,\epsilon) = \left[\mathbb{A}_{0,x}(x,y) + \epsilon \mathbb{A}_{1,x}(x,y)\right] \vec{f}(x,y,\epsilon)$$

$$\partial_y \vec{f}(x,y,\epsilon) = \left[\mathbb{A}_{0,y}(x,y) + \epsilon \mathbb{A}_{1,y}(x,y)\right] \vec{f}(x,y,\epsilon)$$

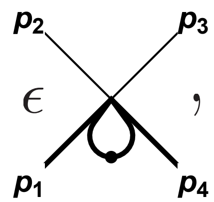
- Use Magnus exponential to find a (matrix) solution at $\epsilon = 0$:

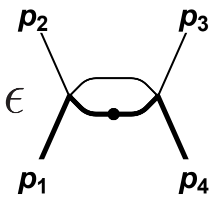
$$\boxed{\begin{aligned} \partial_x \mathbb{B}(x,y) &= \mathbb{A}_{0,x}(x,y)\,\mathbb{B}(x,y) \\ \partial_y \mathbb{B}(x,y) &= \mathbb{A}_{0,y}(x,y)\,\mathbb{B}(x,y) \end{aligned}}$$

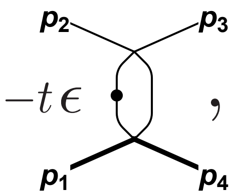
$$\mathbb{B}(s,t,m^2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1/s & 0 & 0 & 0 \\ 0 & 0 & -1/t & 0 & 0 \\ 0 & 0 & 0 & 1/\sqrt{t(t-4m^2)} & 0 \\ 0 & 0 & 0 & 0 & 1/(s-m^2)t \end{pmatrix}$$

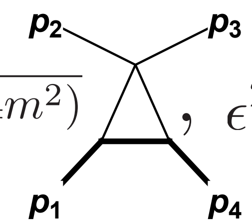
Differential equations II

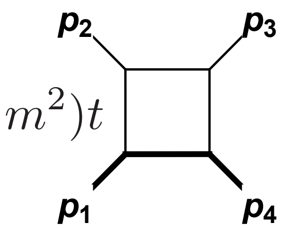
- **New basis** of master integrals

$$\vec{g}(s,t,m^2,\epsilon)=\mathbb{B}^{-1}\vec{f}=\Bigg(\epsilon$$


$$,\ -s\epsilon$$


$$,\ -t\epsilon$$


$$,\ \epsilon\sqrt{t(t-4m^2)}$$


$$,\ \epsilon^2(s-m^2)t$$


$$\Bigg)^T$$

fulfils **canonical** DEQs:

$$\partial_i\vec{g}(x,y,\epsilon)=\epsilon\hat{\mathbb{A}}_1(x,y)\vec{g}(x,y,\epsilon)$$

- The total differential is in **dlog-form**:

$$\boxed{\mathrm{d}\,\vec{g}(x,y,\epsilon)=\epsilon\mathrm{d}\,\mathbb{A}(x,y)\vec{g}(x,y,\epsilon)}$$

$$\begin{aligned}\mathrm{d}\mathbb{A}=&\mathbb{M}_1\mathrm{d}\log(x)+\mathbb{M}_2\mathrm{d}\log(1+x)+\mathbb{M}_3\mathrm{d}\log(y)\\&+\mathbb{M}_4\mathrm{d}\log(1+y)+\mathbb{M}_5\mathrm{d}\log(1-y)\\&+\mathbb{M}_6\mathrm{d}\log(x+y)+\mathbb{M}_7\mathrm{d}\log(1+xy)\end{aligned}$$

$$\mathbb{M}_1=\begin{pmatrix}0&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\quad \mathbb{M}_2=\begin{pmatrix}0&0&0&0&0\\-1&-2&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\2&4&0&0&-2\end{pmatrix},\quad \mathbb{M}_3=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&1&0&0\\1&0&-1&0&0\\1&2&0&0&0\end{pmatrix},\quad \mathbb{M}_4=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&2&0\\0&0&0&0&0\end{pmatrix},$$

$$\mathbb{M}_5=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&-2&0&0\\0&0&0&-2&0\\0&0&2&0&-2\end{pmatrix},\quad \mathbb{M}_6=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\-1&-2&-1&-1&1\end{pmatrix},\quad \mathbb{M}_7=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\-1&-2&-1&1&1\end{pmatrix}$$

Boundary conditions

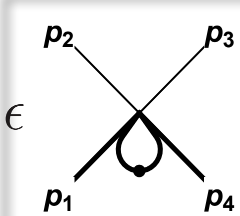
- General solution written in terms of iterated integrals

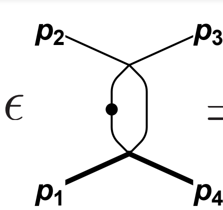
$$\vec{g}(x,y,\epsilon) = \left[\mathbb{1} + \epsilon\int_\gamma \mathrm{d}\mathbb{A} + \epsilon^2 \int_\gamma \mathrm{d}\mathbb{A}\,\mathrm{d}\mathbb{A} + \epsilon^3 \int_\gamma \mathrm{d}\mathbb{A}\,\mathrm{d}\mathbb{A}\,\mathrm{d}\mathbb{A} + \ldots\right]\vec{g}_0(\epsilon)$$

$$\int_\gamma \underbrace{\mathrm{d}\mathbb{A}\ldots\mathrm{d}\mathbb{A}}_{\text{k times}} \Downarrow G(a_1, a_2, \ldots, a_k; z)$$

- **Boundary constants** $\vec{g}_0(\epsilon)$ determined independently

external **input**:

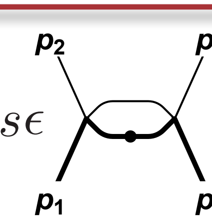
ϵ 

$= 1 \quad -t\,\epsilon$ 

$= \left(\frac{(1-y)^2}{y}\right)^{-\epsilon}\left(1 - \zeta_2\epsilon^2 - 2\zeta_3\epsilon^3 + \mathcal{O}\left(\epsilon^4\right)\right)$

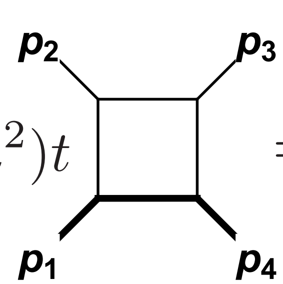
regularity at pseudo-thresholds:

$s \to 0,\ t \to 4m^2,\ s = -t \to m^2/2$

e.g. $\lim_{s\to 0} -s\epsilon$ 

$= 0$

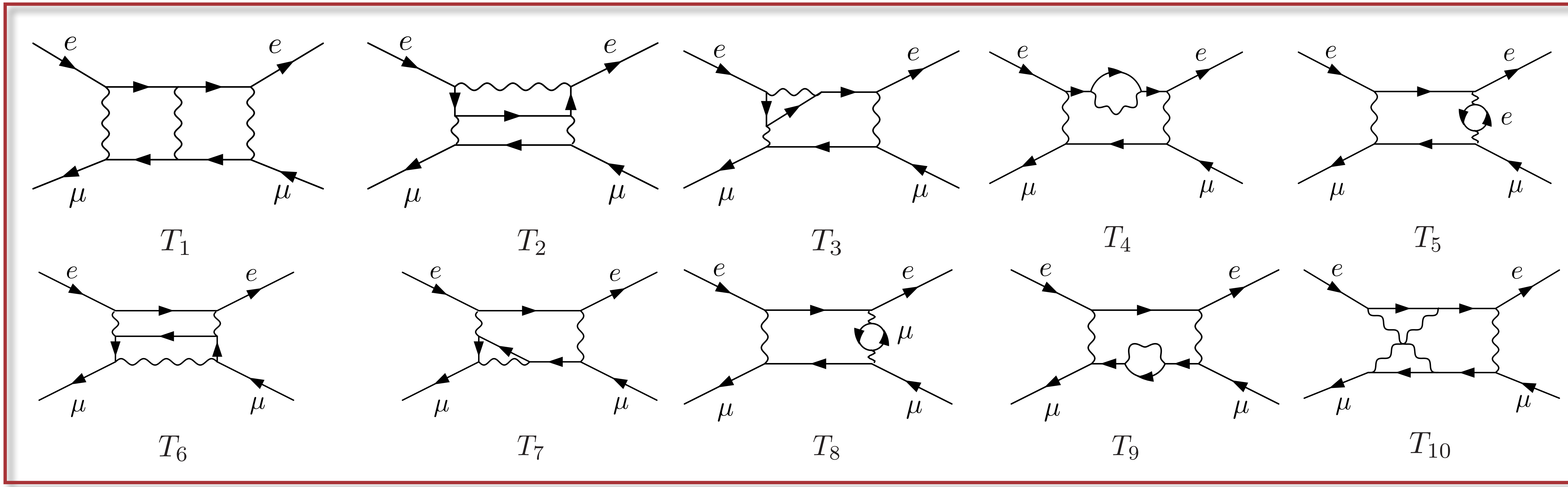
Final result suitable for **analytic continuation** and **numerical evaluation**:

$\epsilon^2(s-m^2)t$ 

$$= 2 + \left(G(-1;x) + G(0;y) - 2G(1;y)\right)\epsilon + \left(-5\zeta_2 + 2G(-1;x)(2G(1;y) - G(0;y))\right)\epsilon^2 + \mathcal{O}(\epsilon^3)$$

μe: two-loop virtual corrections

- **NNLO virtual** corrections: two-loop four-point diagrams



- Add **crossing**-related diagrams and known **vertex** and **self-energy** corrections
- (Part of) these integrals enter:

BhaBha scattering in QED — Gehrmann, Remiddi (01), Bonciani, Mastrolia, Remiddi (04), …

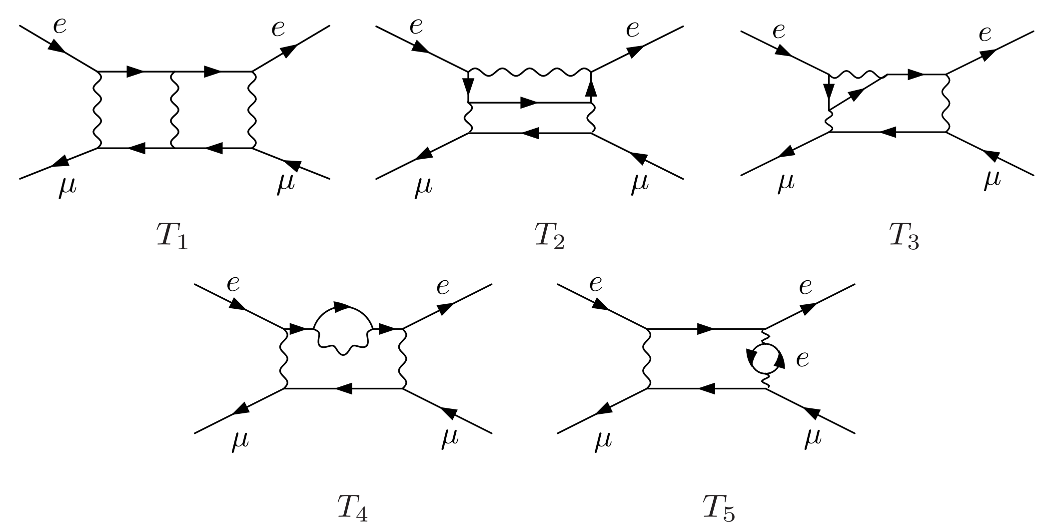
$t\bar{t}$ **production** in QCD — Bonciani, Ferroglia (08), Asatrian, Greub, Pecjak 08, Beneke, Huber, Li (09), …

heavy-to-light quark decay in QCD — Bonciani, Ferroglia Gehrmann et al (08)-(09)-(11)-(13), …

Two-loop planar integrals

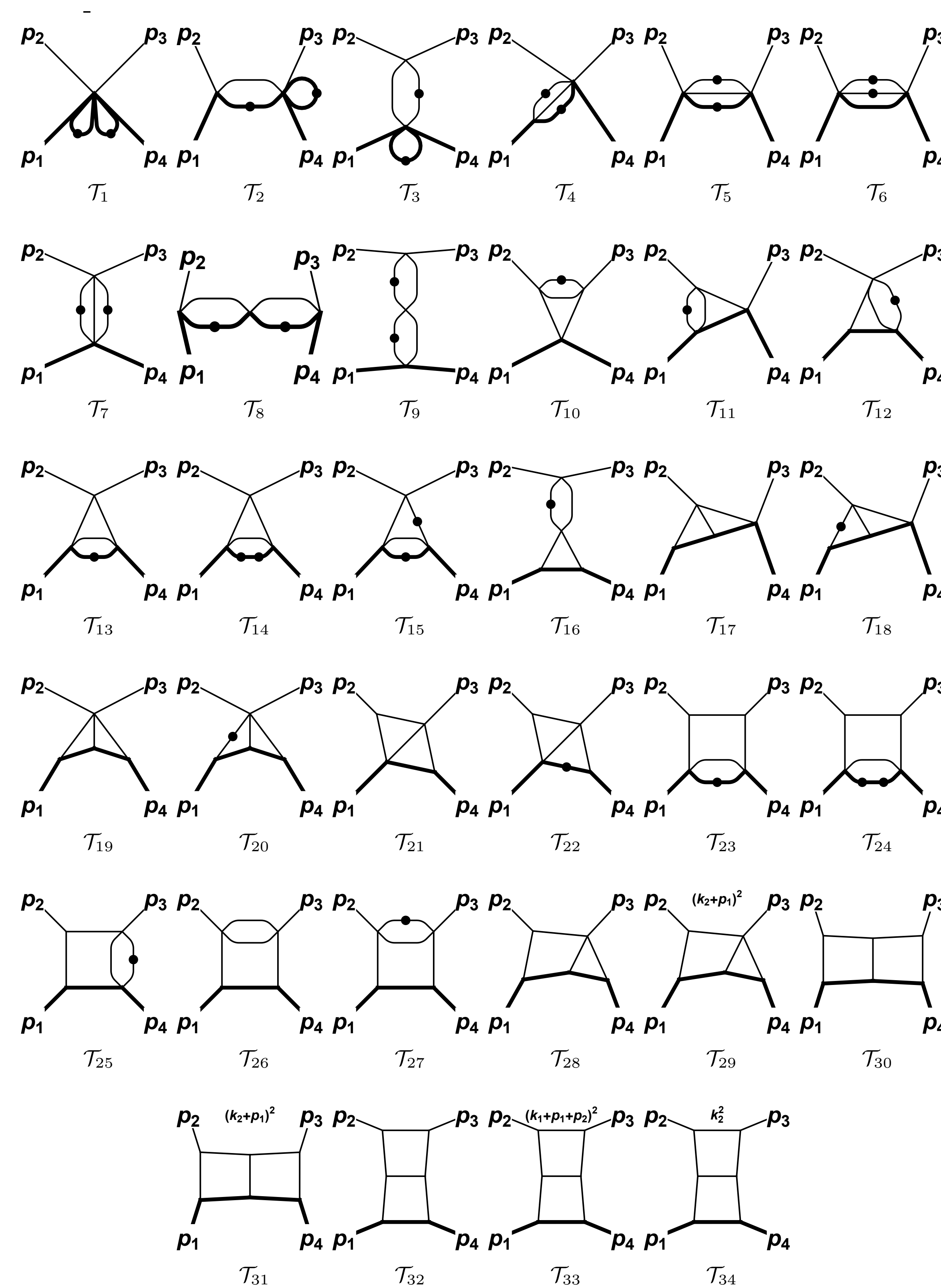
Mastrolia, Passera, AP, Schubert 17

- T_1- T_5 define the **integral family F1**



$$\int \widetilde{\mathrm{d}^d k_1} \widetilde{\mathrm{d}^d k_2} \frac{1}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5} D_6^{n_6} D_7^{n_7} D_8^{n_8} D_9^{n_9}}, \quad n_i \in \mathbb{Z}$$

$$D_1 = k_1^2 - m^2, \quad D_2 = k_2^2 - m^2, \quad D_3 = (k_1 + p_1)^2, \quad D_4 = (k_2 + p_1)^2,$$
$$D_5 = (k_1 + p_1 + p_2)^2, \quad D_6 = (k_2 + p_1 + p_2)^2, \quad D_7 = (k_1 - k_2)^2,$$
$$D_8 = (k_1 + p_4)^2, \quad D_9 = (k_2 + p_4)^2 \quad .$$

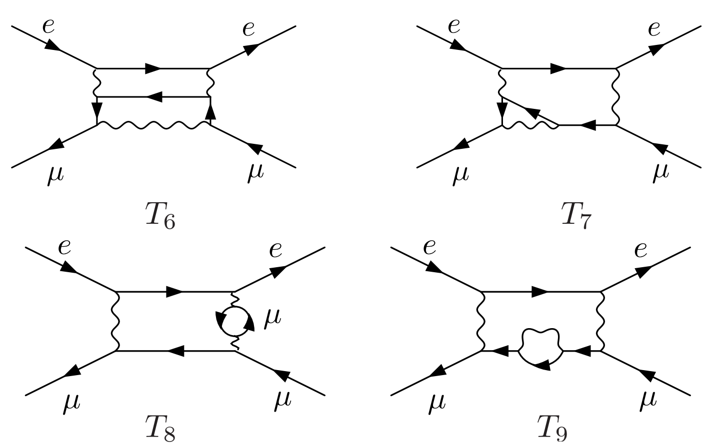


- **34 master integrals** identified from IBP's

Two-loop planar integrals

Mastrolia, Passera, AP, Schubert 17

- T_6 - T_9 define the **integral family F2**

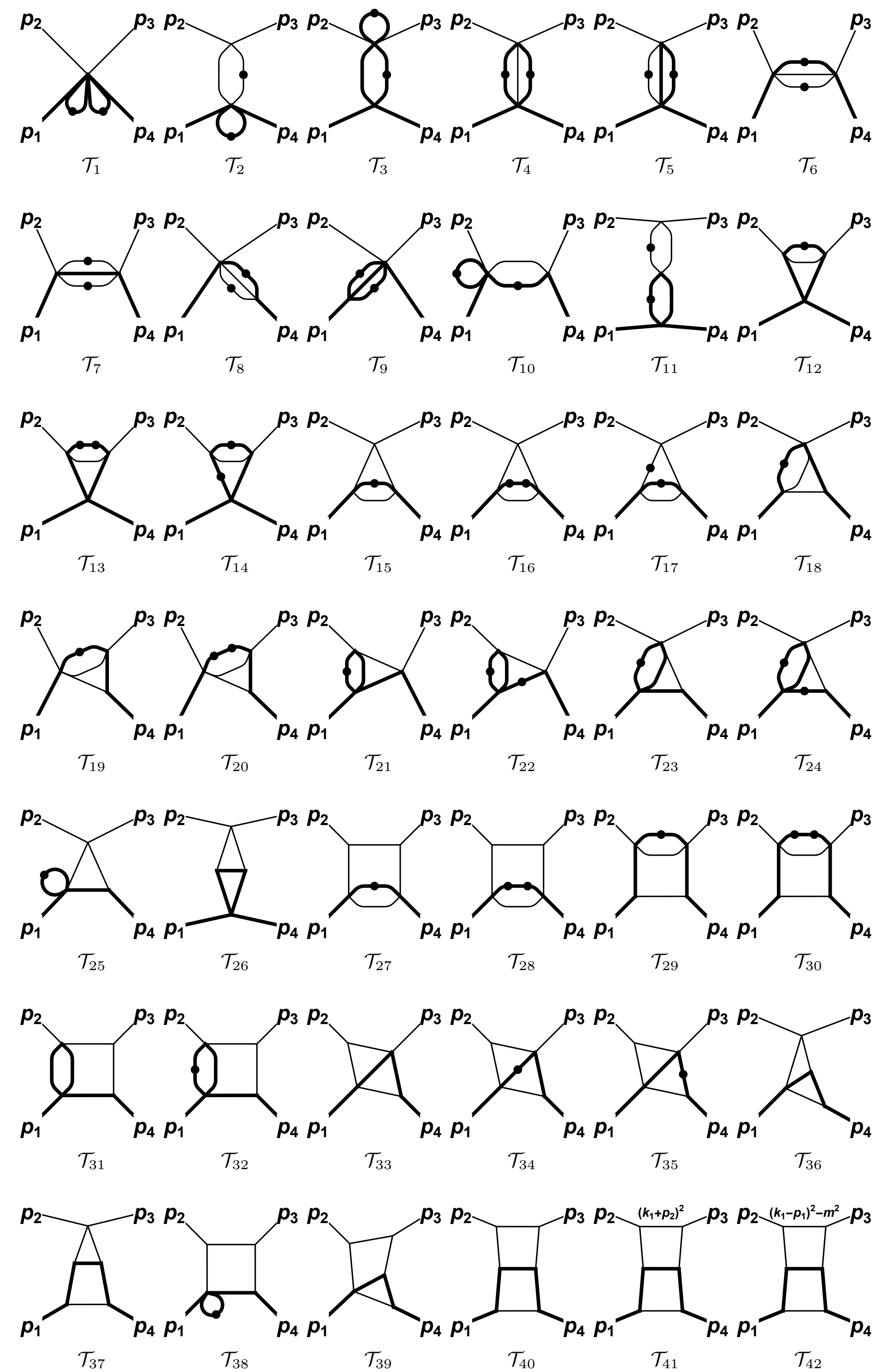


$$\int \widetilde{\mathrm{d}^d k_1} \widetilde{\mathrm{d}^d k_2} \, \frac{1}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5} D_6^{n_6} D_7^{n_7} D_8^{n_8} D_9^{n_9}} \,, \qquad n_i \in \mathbb{Z}$$

$$D_1 = k_1^2 - m^2, \quad D_2 = k_2^2 - m^2, \quad D_3 = (k_1 + p_1)^2, \quad D_4 = (k_2 + p_1 + p_2)^2,$$
$$D_5 = (k_1 - k_2)^2, \quad D_6 = (k_2 + p_1)^2, \quad D_7 = (k_1 + p_1 + p_2)^2,$$

- **42 master integrals** identified from IBP's

- F1 and F2 have **65 distinct MIs**



Differential equations

- Identify a set of **pre-canonical** integrals for F1 and F2

$$\partial_x \vec{f}(x,y,\epsilon) = \left[\mathbb{A}_{0,x}(x,y) + \epsilon \mathbb{A}_{1,x}(x,y)\right]\vec{f}(x,y,\epsilon)$$
$$\partial_y \vec{f}(x,y,\epsilon) = \left[\mathbb{A}_{0,y}(x,y) + \epsilon \mathbb{A}_{1,y}(x,y)\right]\vec{f}(x,y,\epsilon)$$

$$\boxed{-\frac{s}{m^2} = x\,, \qquad -\frac{t}{m^2} = \frac{(1-y)^2}{y}}$$

- Use **Magnus** exponential to obtain a **canonical** system of DEQs

$$\boxed{\mathrm{d}\,\vec{g}(x,y,\epsilon) = \epsilon \mathrm{d}\,\mathbb{A}(x,y)\vec{g}(x,y,\epsilon)}$$

$$\boxed{\begin{aligned}\mathrm{d}\mathbb{A} =& \mathbb{M}_1 \mathrm{d}\log(x) + \mathbb{M}_2 \mathrm{d}\log(1+x) + \mathbb{M}_3 \mathrm{d}\log(1-x) + \mathbb{M}_4 \mathrm{d}\log(y)\\ &+ \mathbb{M}_5 \mathrm{d}\log(1+y) + \mathbb{M}_6 \mathrm{d}\log(1-y) + \mathbb{M}_7 \mathrm{d}\log(x+y)\\ &+ \mathbb{M}_8 \mathrm{d}\log(1+xy) + \mathbb{M}_9 \mathrm{d}\log(1-y-xy+y^2)\end{aligned}}$$

- Integrate the DEQs in terms of **GPLs**

- Fix **boundary conditions**

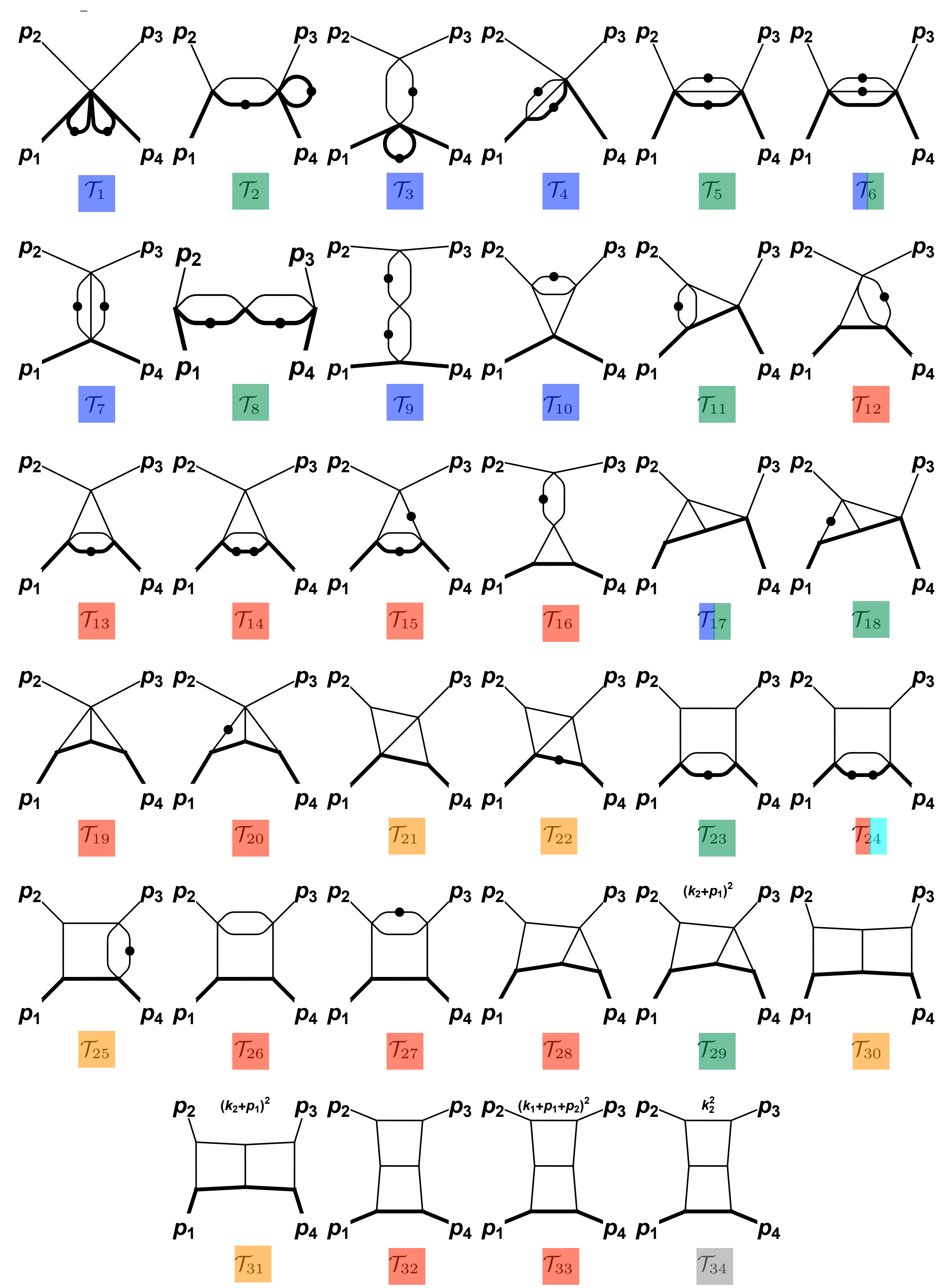
Boundary conditions

- Boundary for **F1** given by **8 input** integrals and regularity at **pseudo-thresholds** :

  - Input integrals
  - Regularity at $s \to 0$
  - Regularity at $t \to 4m^2$
  - Regularity at $u \to 2m^2$
  - Regularity at $u \to \infty$
  - Regularity at $s \to -m^2$

- Combinations of constant GPLs fitted to ζ_k (PSLQ):

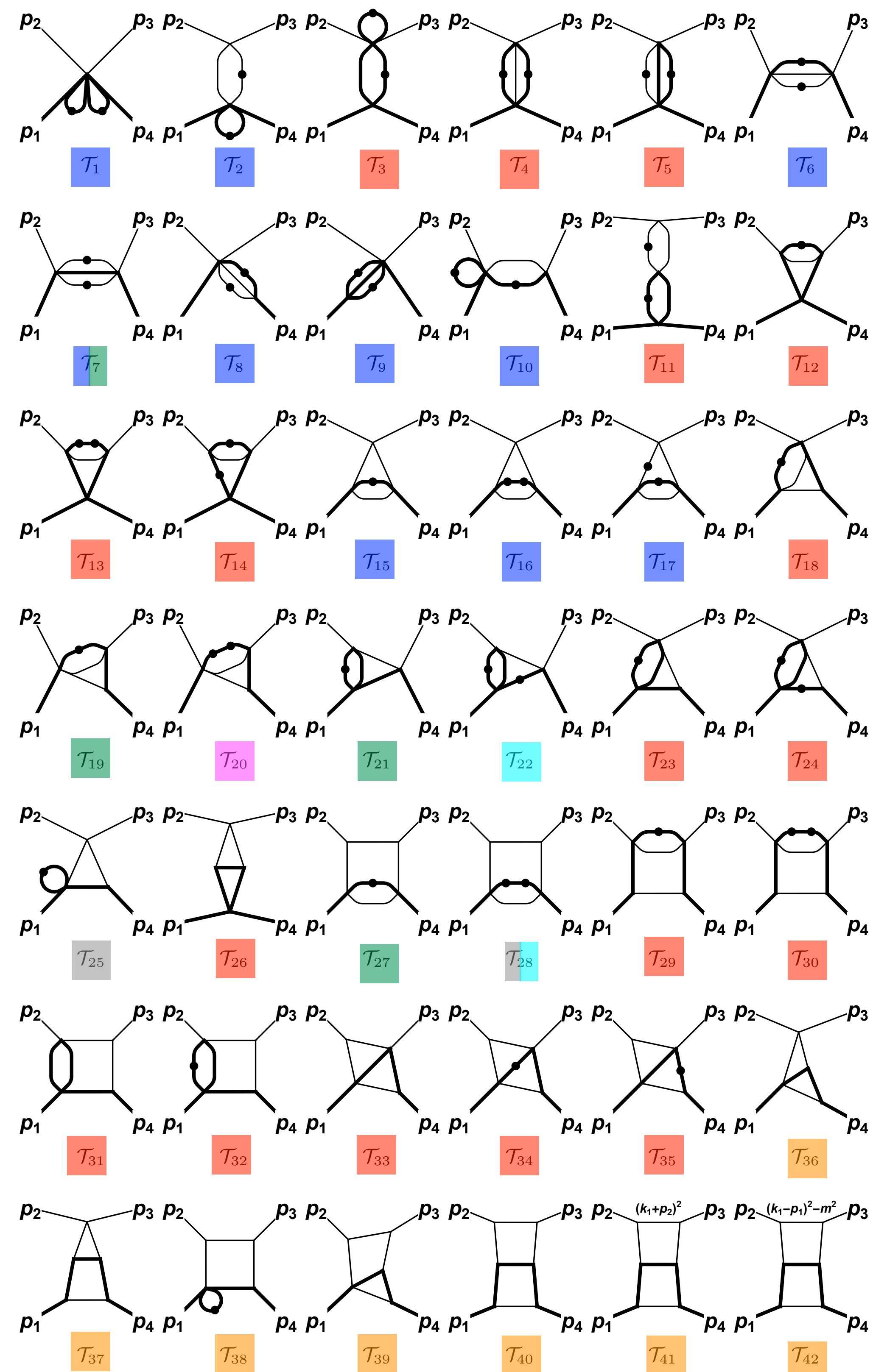
$$\boxed{\begin{aligned}-59\zeta_4 = \pi^2 \Big(&G(-1;1)^2 - 2\,G(0,-(-1)^{\frac{1}{3}};1) - 2\,G(0,(-1)^{\frac{2}{3}};1)\Big) - 21\,\zeta_3\,G(-1;1)\\ &-G(-1;1)^4 - 18\,G(0,0,0,-(-1)^{\frac{1}{3}};1) - 18\,G(0,0,0,(-1)^{\frac{2}{3}};1)\\ &+12\,G(0,0,-(-1)^{\frac{1}{3}},-1;1) + 12\,G(0,0,(-1)^{\frac{2}{3}},-1;1)\\ &+12\,G(0,-(-1)^{\frac{1}{3}},-1,-1;1) + 12\,G(0,(-1)^{\frac{2}{3}},-1,-1;1) + 24\,G(0,0,0,2;1)\end{aligned}}$$



Boundary conditions

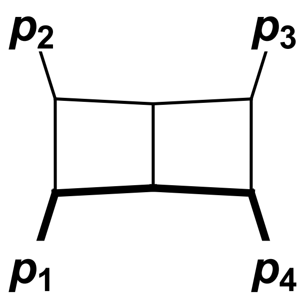
- Boundary for **F2** given by **9 input** integrals and regularity at **pseudo-thresholds** :

  - Input integrals
  - Regularity at $s \to 0$
  - Regularity at $t \to 0$
  - Regularity at $u \to m^2/2$
  - Regularity at $s \to -m^2$
  - Regularity at $t \to 4m^2$
  - Regularity at $s \to 2t - m^2 - \sqrt{-t}\sqrt{4m^2 - t/2}$

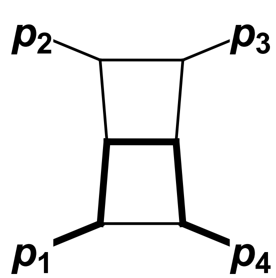


Numerical evaluation

- **Numerical evaluation** with GiNaC validated against SecDeC
- **Euclidean kinematics :** $\boxed{m^2 = 1,\ s = -1/2,\ t = -4/3}$

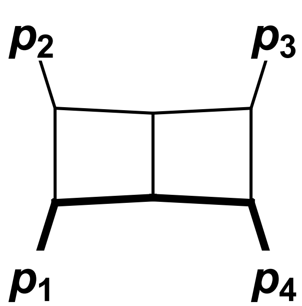


$$= -\frac{4}{9\epsilon^4} + \frac{0.9722833780002695}{\epsilon^3} + \frac{2.2577379083718387}{\epsilon^2} - \frac{0.6831983770352587}{\epsilon} - 16.738573400402935 + \mathcal{O}(\epsilon)$$

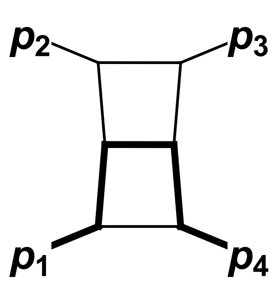


$$= \frac{0.30898470618790586}{\epsilon^3} - \frac{0.13117172678776337}{\epsilon^2} - \frac{1.6429065516247423}{\epsilon} - 6.8582633086260625 + \mathcal{O}(\epsilon)$$

-**Physical region :** $\boxed{m^2 = 1,\ s = 5/2,\ t = -1/2}$



$$= -\frac{32}{27\epsilon^4} + \frac{1.2849833364496313 - i6.515895874145223}{\epsilon^3} + \frac{14.616286362308317 + i1.2893222586216164}{\epsilon^2} + \frac{3.7044754877481445 + i19.653338134527136}{\epsilon} + (31.933312628027124 + i51.70714567127463) + \mathcal{O}(\epsilon)$$

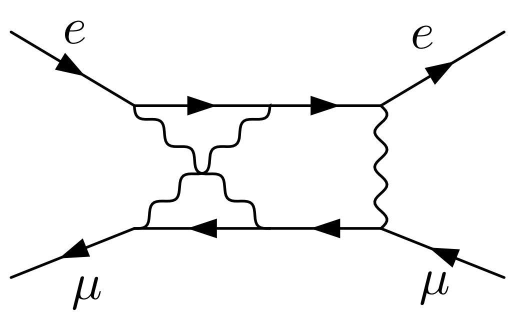


$$= -\frac{0.9241962407465936}{\epsilon^3} - \frac{0.20971796688521188 + i7.742528320901192}{\epsilon^2} + \frac{24.805385023064197 - i9.913936271553196}{\epsilon} + (41.399867536999416 + i64.54738285675748) + \mathcal{O}(\epsilon)$$

μe: two-loop non-planar integrals

- A single **non-planar** integral family is missing :

Mastrolia, AP, Schubert, *in progress*

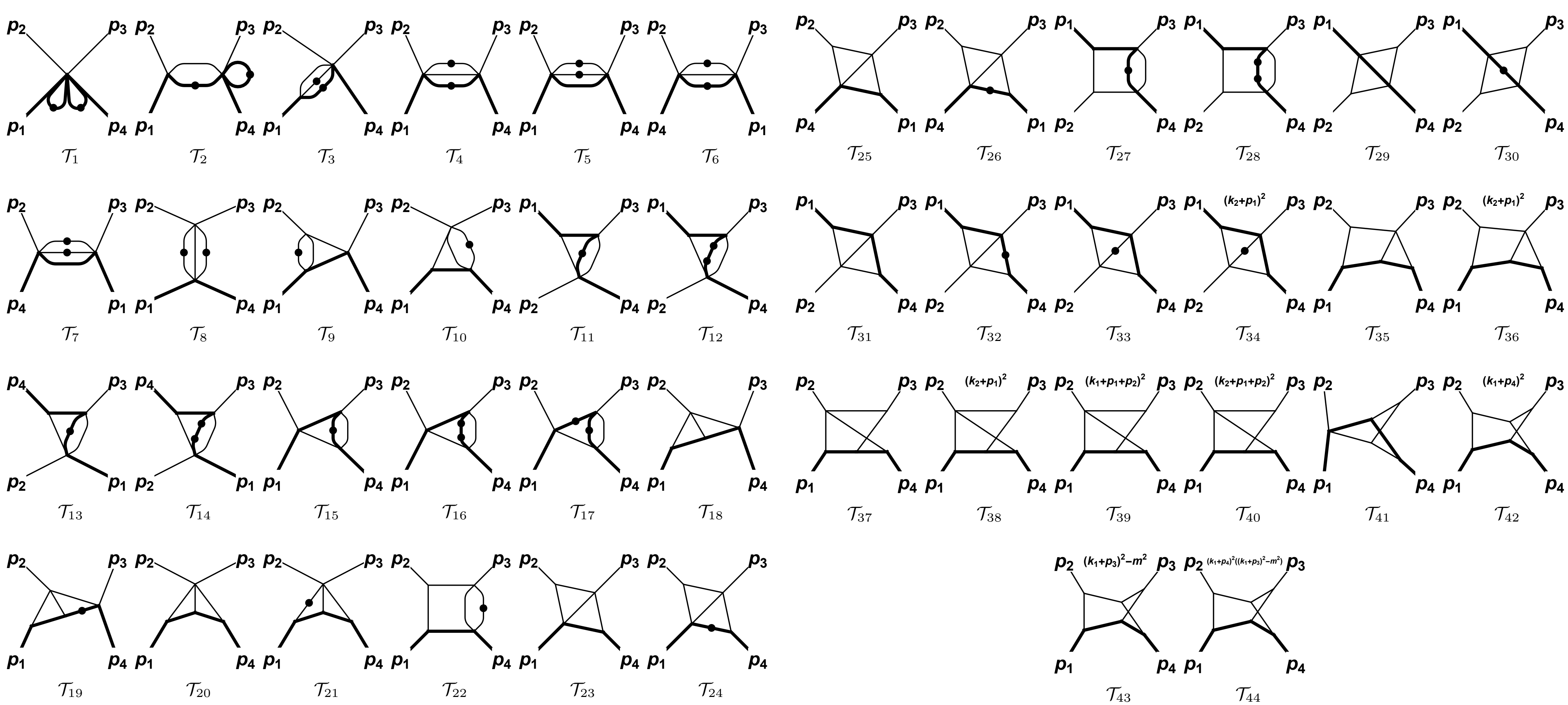


T_{10}

$$\int \widetilde{\mathrm{d}^d k_1}\,\widetilde{\mathrm{d}^d k_2}\,\frac{1}{D_1^{n_1}D_2^{n_2}D_3^{n_3}D_4^{n_4}D_5^{n_5}D_6^{n_6}D_7^{n_7}D_8^{n_8}D_9^{n_9}}\,, \qquad n_i \in \mathbb{Z}$$

$$D_1 = (k_1)^2 - m^2, \quad D_2 = (k_2)^2 - m^2, \quad D_3 = (k_1+p_1)^2, \quad D_4 = (k_2+p_1)^2,$$
$$D_5 = (k_1+p_1+p_2)^2, \quad D_6 = (k_2+p_1+p_2)^2, \quad D_7 = (k_1-k_2)^2,$$
$$D_8 = (k_2+p_1+p_2+p_3)^2, \quad D_9 = (k_1-k_2+p_3)^2\,,$$

- **44 master integrals** identified from IBP’s



Differential equations

- **Pre-canonical** basis has been identified

$$\partial_z \vec{f}(z,y,\epsilon) = \left[\mathbb{A}_{0,z}(z,y) + \epsilon \mathbb{A}_{1,z}(z,y)\right] \vec{f}(z,y,\epsilon)$$
$$\partial_y \vec{f}(z,y,\epsilon) = \left[\mathbb{A}_{0,y}(z,y) + \epsilon \mathbb{A}_{1,y}(z,y)\right] \vec{f}(z,y,\epsilon)$$

$$\boxed{\frac{s}{m^2} = 1 + \frac{(1-y)^2}{y-z^2} \qquad -\frac{t}{m^2} = \frac{(1-y)^2}{y}}$$

- **Canonical** system of DEQs achieved through Magnus method

$$\mathrm{d}\vec{g}(z,y,\epsilon) = \epsilon \mathrm{d}\, \mathbb{A}(z,y) \vec{g}(z,y,\epsilon)$$

- The alphabet contains **12 polynomial letters**

$$\boxed{\begin{aligned} \mathrm{d}\mathbb{A} =& \mathbb{M}_1 \mathrm{d}\log(y) + \mathbb{M}_2 \mathrm{d}\log(1+y) + \mathbb{M}_3 \mathrm{d}\log(1-y) + \mathbb{M}_4 \mathrm{d}\log(x) + \mathbb{M}_5 \mathrm{d}\log(1+x) \\ &+ \mathbb{M}_6 \mathrm{d}\log(1-x) + \mathbb{M}_7 \mathrm{d}\log(y+z) + \mathbb{M}_8 \mathrm{d}\log(y-z) + \mathbb{M}_9 \mathrm{d}\log(y-z^2) \\ &+ \mathbb{M}_{10} \mathrm{d}\log(1-y+y^2-z^2) + \mathbb{M}_{11} \mathrm{d}\log(1-3y+y^2+z^2) + \mathbb{M}_{12} \mathrm{d}\log(y^2-z^2+yz^2-y^2z^2) \end{aligned}}$$

- General solution expressed in terms of **GPLs**

- Ongoing boundary fixing

Conclusions

- The computation of the NNLO virtual corrections to μe-scattering requires the evaluation of previously unknown master integrals

- Canonical system of DEQs for all masters integrals obtained through the Magnus exponential method

- All integrals written in terms of generalised polylogarithms

- Boundary constants fixed by demanding the regularity of the master integrals at pseudo-thresholds

- All planar integrals have been computed and checked against SecDec

Outlook

- The non-planar integrals will be completed soon

  - Crosscheck with recent numerical determination in $\eta_c \to \mathrm{had}$ **Liu, Ma,Wang 17**

- Master integrals plugged in the amplitude decomposition to get analytical expression of $\mathcal{M}^{(2)}$ (+ renormalization) **see Torres' talk**

- $|\mathcal{M}_\gamma^{(1)}|^2$ and real-virtual contributions can be computed with GoSam **see Greiner's talk**

- All ingredient must be combined within a subtraction framework ($m_e \neq 0$ effects?)