Tests of QED with the bound electron g-factor

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MITP topical workshop
"The Evaluation of the Leading Hadronic Contribution to the Muon Anomalous Magnetic Moment"

Outline

- Tests of the Standard Model with bound states
- Measurements of the bound electron g-factor and determination of the electron mass
- Bound electron g-factor – theoretical perspective
- Future perspective: determination of α, bound muon g-factor?

Electron g-2

Experiment:

$$a_e = 1\,159\,652\,180.73(0.28) \times 10^{-12}$$

[D. Hanneke, S. Fogwell Hoogerheide, and G. Gabrielse, Phys.Rev.A 83, 052122 (2011)]

Theory:

$$a_e = 1\,159\,652\,181.78(0.77) \times 10^{-12}$$

[T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio, Phys.Rev.Lett. 109, 111807 (2012)]

The theory error is dominated by α

$$\alpha^{-1}(Rb) = 137.035\,999\,037(91)$$

so we use a_e to determine α

$$\alpha^{-1}(a_e) = 137.035\,999\,139(31)$$

a_e is a yardstick of modern physics!

Can we use electron to test muon g-2?

Electron g-2 may be sensitive to the same New Physics

$$\Delta a_e \sim \frac{m_e^2}{m_\mu^2} \Delta a_\mu \sim 7 \times 10^{-14}$$

Improvement by a factor of 4 with respect to existing measurement is needed.

Additionally, a new source of α is needed

- Atomic spectroscopy $R_\infty = \dfrac{\alpha^2 m_e c}{4\pi\hbar}$

Why spectroscopy?

Spectroscopic measurements of transition frequencies have typically very good precision.

1) Extract R_∞ from the data $\nu_{ij} = \varepsilon_j - \varepsilon_i$

$$\varepsilon_i = -\frac{R_\infty c}{n_i^2}(1 + \delta_i)$$

2) Determine $\frac{\hbar}{m_e}$

$$\frac{\hbar}{m_e} = \frac{u}{m_e}\frac{M_X}{u}\frac{\hbar}{M_X}$$

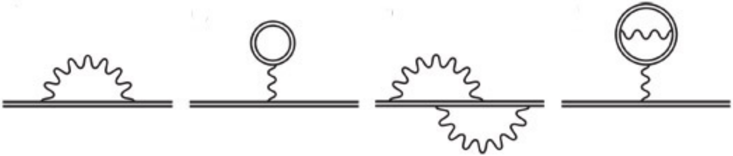
$\frac{u}{m_e}$ determined from the g-factor of a bound electron

$\frac{\hbar}{M_X}$ - recoil velocity of Rb atom $v_r = \frac{\hbar k}{M_{\mathrm{Rb}}}$ [R. Bouchendira et al. Phys.Rev.Lett.106:080801,2011] *currently the limiting factor!*

Higher order corrections

We need to know δ_i which contains

- Higher order corrections



- Recoil corrections $\sim \dfrac{m_e}{m_p}$
- Nuclear size and structure corrections

$$\Delta E = \frac{2\pi}{3}(Z\alpha)\langle r_p^2\rangle \left|\psi(0)\right|^2$$

Higher order corrections

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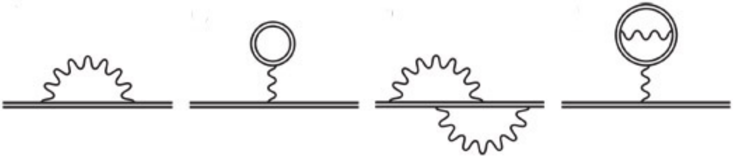
- Higher order corrections



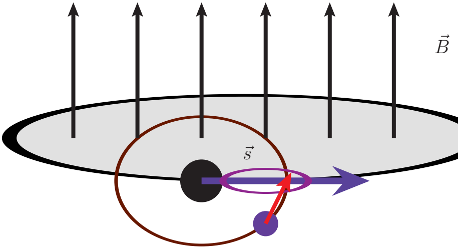
- Recoil corrections $\sim \dfrac{m_e}{m_p}$
- Nuclear size and structure corrections

$$\Delta E = \frac{2\pi}{3}(Z\alpha)\langle r_p^2\rangle \left|\psi(0)\right|^2$$

But what if we measure the spectrum in the presence of magnetic field?

Bound electron g-factor

A hydrogen-like ion in a magnetic field



1S state, zero-spin nuclei

Spin precession: $\omega_L = \frac{g}{2}\frac{e}{m_e}B$

Ion motion: $\omega_c = \frac{Q}{M}B,\ Q = (Z-1)e$

What about $\frac{m_e}{u}$?

The simplest case is to take hydrogen-like ions.

Larmor frequency: $\omega_L = \frac{g}{2}\frac{e}{m_e}B$

Cyclotron frequency of the ion: $\omega_c = \frac{Q}{M}B,\ Q = (Z-1)e$

$$m_e = \frac{g}{2}\frac{e}{Q}\frac{\omega_c}{\omega_L}M$$

For nuclei with a well known mass (e.g. Carbon) – best source of $\frac{m_e}{u}$ assuming the correctness of theoretical prediction for g-factor

Theory of bound electron g-factor

$$
\begin{aligned}
g &= g(Z\alpha,\alpha)\\
&= g(Z\alpha,0)+\frac{\alpha}{\pi}A(Z\alpha)+\left(\frac{\alpha}{\pi}\right)^2 B(Z\alpha)+\ldots
\end{aligned}
$$

For large Z: direct evaluation in the Furry picture, i.e. no expansion in $Z\alpha$; numerical results available only for one-loop and a certain class of diagrams at two-loop level. [see e.g. V.A. Yerokhin, Z. Harman Phys.Rev. A95, 060501, 2017; V.A. Yerokhin, Z. Harman, Phys.Rev. A88, 042502, 2013]

For small Z: expansion in $Z\alpha$ with the help of modern EFT methods (NRQED, PNRQED)

Theory of bound electron g-factor

$$
\begin{aligned}
g &= g(Z\alpha, \alpha) \\
&= g(Z\alpha, 0) + \frac{\alpha}{\pi} A(Z\alpha) + \left(\frac{\alpha}{\pi}\right)^2 B(Z\alpha) + \ldots
\end{aligned}
$$

For small Z: expansion in $Z\alpha$ with the help of modern EFT methods (NRQED, PNRQED)

$$
\begin{aligned}
A(Z\alpha) &= A_{20}(Z\alpha)^2 + A_{41}(Z\alpha)^4 \ln(Z\alpha) + A_{40}(Z\alpha)^4 + A_{50}(Z\alpha)^5 + \ldots \\
B(Z\alpha) &= B_{20}(Z\alpha)^2 + B_{41}(Z\alpha)^4 \ln(Z\alpha) + B_{40}(Z\alpha)^4 + B_{50}(Z\alpha)^5 + \ldots
\end{aligned}
$$

Computations usually involve sum over infinite number of Feynman diagrams – exchange of potential photons does not generate suppression and has to be resummed

Theory of bound electron g-factor

$$g = g(Z\alpha, \alpha)$$
$$= g(Z\alpha, 0) + \frac{\alpha}{\pi}A(Z\alpha) + \left(\frac{\alpha}{\pi}\right)^2 B(Z\alpha) + \dots$$

We also need

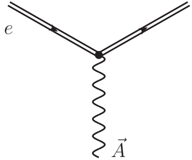
- Recoil corrections $\sim \frac{m_e}{m_N}$ [V. M. Shabaev and V. A. Yerokhin, Phys.Rev.Lett. 88, 091801, 2002; K. Pachucki, Phys.Rev. A78 , 012504, 2008.]
- Finite nuclear size corrections $\sim \frac{r_N}{r_B}$, $r_B = \frac{1}{m_N Z\alpha}$ [S. G. Karshenboim, Phys.Lett. A266, 380, 2000; S. G. Karshenboim, V, G. Ivanov, Phys.Rev. A97, 022506, 2018]

$g(Z\alpha, 0)$

Leading effect [Breit, Nature, 1928]

$$\Delta E \sim \left\langle \vec{\alpha} \cdot \vec{A} \right\rangle_{1S}$$

Computation is like for a free particle but external states are Dirac Hydrogen wave-functions



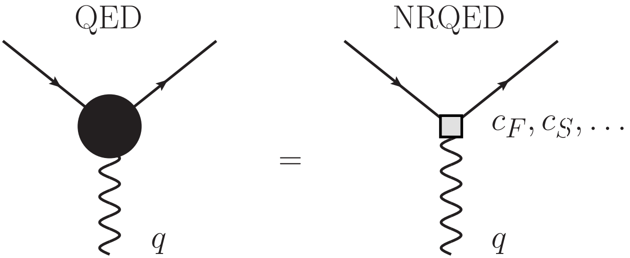
$$g_e = \frac{2}{3}\left(1 + 2\sqrt{1-(Z\alpha)^2}\right) \approx 2 - \frac{2}{3}(Z\alpha)^2$$

A_{20} and B_{20}

First corrections are universal

$$\begin{aligned}\Delta H &= -c_F \frac{e}{2m}\vec{\sigma}\cdot\vec{B} - i c_S \frac{e}{8m^2}\vec{\sigma}\cdot\left(\vec{D}\times\vec{E} - \vec{E}\times\vec{D}\right) \\ &- c_{W1}\frac{e}{8m^3}\left\{\vec{D}^2, \vec{\sigma}\cdot\vec{B}\right\} + c_{W2}\frac{e}{4m^3} D^i \vec{\sigma}\cdot\vec{B} D_i \\ &- c_{p'p}\frac{e}{8m^3}\left(\vec{\sigma}\cdot\vec{D}\,\vec{B}\cdot\vec{D} + \vec{D}\cdot\vec{B}\,\vec{\sigma}\cdot\vec{D}\right)\end{aligned}$$

where the matching coefficients depend only on the Dirac and Pauli form-factors at $q^2 = 0$.



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First corrections are universal

$$\Delta H = -c_F \frac{e}{2m}\vec{\sigma}\cdot\vec{B} - ic_S\frac{e}{8m^2}\vec{\sigma}\cdot\left(\vec{D}\times\vec{E}-\vec{E}\times\vec{D}\right) - c_{W1}\frac{e}{8m^3}\left\{\vec{D}^2,\vec{\sigma}\cdot\vec{B}\right\} + c_{W2}\frac{e}{4m^3}D^i\vec{\sigma}\cdot\vec{B}D_i - c_{p'p}\frac{e}{8m^3}\left(\vec{\sigma}\cdot\vec{D}\,\vec{B}\cdot\vec{D}+\vec{D}\cdot\vec{B}\,\vec{\sigma}\cdot\vec{D}\right)$$

where the matching coefficients depend only on the Dirac and Pauli form-factors at $q^2 = 0$.

Hence, this terms can be expressed in terms of the free electron g-factor

$$\frac{g_e}{2} = 1 - \frac{(Z\alpha)^2}{3} + F_P(0)\left(1+\frac{(Z\alpha)^2}{6}\right) \quad (1)$$

which is universal and valid to all orders in $\frac{\alpha}{\pi}$

[H. Grotch, Phys. Rev. A 2, 1605, 1970; A. Czarnecki, K. Melnikov, and A. Yelkhovsky Phys. Rev. A 63, 012509, 2000]

Vacuum polarization and diagrams with closed fermionic loop

Analytical results for one- and two-loop diagrams up to the order $(Z\alpha)^5$ [U.D. Jentschura, Phys.Rev. A79, 044501, 2009]
Numerical results for two-loop diagrams with a closed fermionic loop [V. A. Yerokhin, Z. Harman, Phys.Rev. A 88, 042502, 2013]

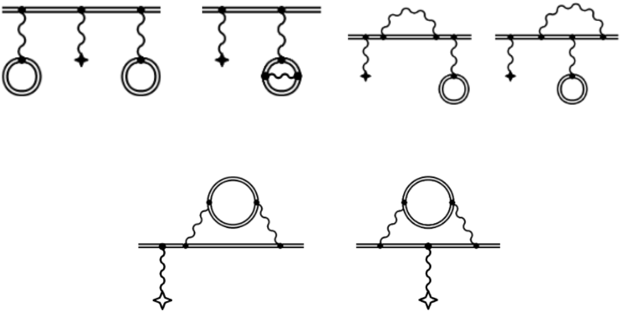


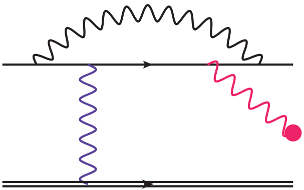
Image credit: Phys.Rev. A 88, 042502, 2013

A_{41}, A_{40}, B_{41} and B_{40}

Self-energy and vacuum polarization

$\mathcal{O}\left(\alpha(Z\alpha)^4\right)$: [K. Pachucki, U. Jentschura, and V. A. Yerokhin, Phys.Rev.Lett. 93, 150401, 2004]

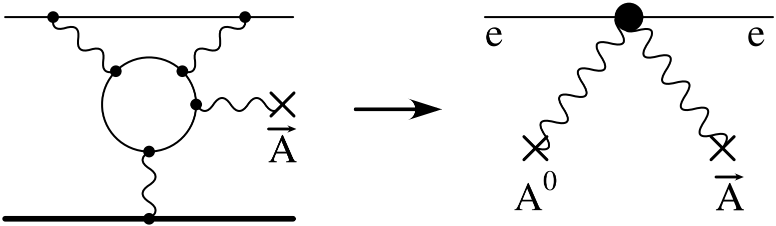
$\mathcal{O}\left(\alpha^2(Z\alpha)^4\right)$: [K. Pachucki, A. Czarnecki, U. Jentschura, and V.A. Yerokhin, Phys.Rev. A 72, 022108, 2005]



$$\begin{aligned} g_e^{(2,4)} &= \left(\frac{\alpha}{\pi}\right)^2 \frac{(Z\,\alpha)^4}{n^3}\left\{\frac{28}{9}\ln[(Z\,\alpha)^{-2}] + \frac{258917}{19440} - \frac{4}{9}\ln k_0 - \frac{8}{3}\ln k_3 + \frac{113}{810}\pi^2 - \frac{379}{90}\pi^2\ln 2 + \frac{379}{60}\zeta(3)\right. \\ &+ \left.\left(\frac{16-19\pi^2}{108}\right)_{\mathrm{LBL}} + \frac{1}{n}\left[-\frac{985}{1728} - \frac{5}{144}\pi^2 + \frac{5}{24}\pi^2\ln 2 - \frac{5}{16}\zeta(3)\right]\right\} \end{aligned}$$

B_{40}: LBL correction

Calculation of the LBL correction to the bound electron g-2 is similar to Lamb



$$\mathcal{L}_{\mathrm{NRQED}} \supset \frac{\psi^\dagger(\vec{\sigma}\cdot\vec{B})(\vec{\nabla}\cdot\vec{E})\psi}{m_e^3}$$

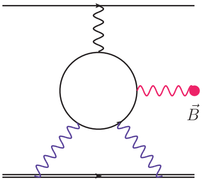
The LBL correction (not included in previous evaluation of $(Z\alpha)^4\left(\frac{\alpha}{\pi}\right)^2$ terms)

$$\delta g_e = (Z\alpha)^4\left(\frac{\alpha}{\pi}\right)^2\frac{16-19\pi^2}{108}$$

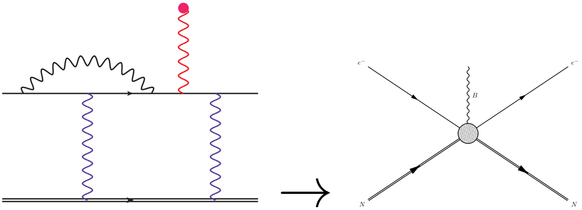
[A. Czarnecki, R.S., Phys.Rev. A94, 060501, 2016]

A_{50}

LBL [S.G. Karshenboim and A.I. Milstein, PLB 549, 321, 2002]



Self-energy [K. Pachucki, M. Puchalski, Phys.Rev. A96, 032503, 2017]

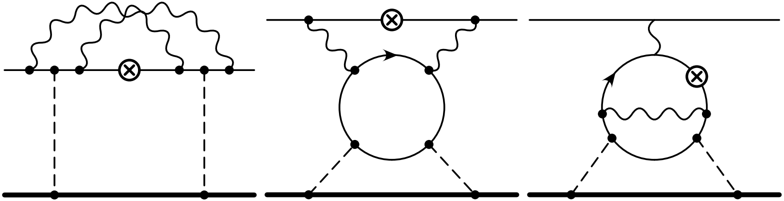


Two loop correction - three loop computation

- self-energy
- LBL
- magnetic loop

- Over 100 diagrams
- 32 master integrals

[A. Czarnecki, M. Dowling, J. Piclum, R.S., Phys.Rev.Lett. 120, 043203, 2018]



Contribution	$^4\mathrm{He}^+$	$^{12}\mathrm{C}^{5+}$	$^{28}\mathrm{Si}^{13+}$
Dirac/Breit value	1.999 857 988 825 37(7)	1.998 721 354 392 0(6)	1.993 023 571 557(3)
+ other known corrections	2.002 177 406 711 41(55)	2.001 041 590 168 6(12)	1.995 348 957 825 (39)
g^{SE}	0.000 000 000 000 02	0.000 000 000 005 0	0.000 000 000 348
g^{LBL}	-0.000 000 000 000 01	-0.000 000 000 001 5	-0.000 000 000 102
g^{ML}	0.000 000 000 000 00	0.000 000 000 000 6	0.000 000 000 038
H.O.	0.000 000 000 000 00(3)	0.000 000 000 000 0(93)	0.000 000 000 000(590)
Total	2.002 177 406 711 42(55)	2.001 041 590 172 7(94)	1.995 348 958 109 (591)

Experiment vs. Theory

RAPID COMMUNICATIONS

PHYSICAL REVIEW A **87**, 030501(R) (2013)

g-factor measurement of hydrogenlike $^{28}Si^{13+}$ as a challenge to QED calculations

S. Sturm,[1,2] A. Wagner,[1] M. Kretzschmar,[2] W. Quint,[3] G. Werth,[2] and K. Blaum[1]

[1]*Max-Planck-Institut für Kernphysik, Saupfercheckweg 1, 69117 Heidelberg, Germany*

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(Received 30 May 2012; published 8 March 2013)

For $^{28}\mathrm{Si}^{13+}$ [S. Sturm, A. Wagner, M. Kretzschmar, W. Quint, G. Werth, and K. Blaum, Phys. Rev. A 87, 030501, 2013]:

$$
\begin{aligned}
g_{exp} &= 1.995\,348\,959\,10(7)_{stat}(7)_{syst}(80)_{m_e} \\
g_{th} &= 1.995\,348\,958\,11(59)
\end{aligned}
$$

Use g-factor to determine m_e just like $g-2$ is used to determine α!

Electron mass

Combination of measurements for Carbon and Silicon results in improvement by a factor of 13 compared to previous CODATA value.

$$m_e = 0.000\,548\,579\,909\,065(16)u$$

[S. Sturm et al. Nature 506, 467, 2014;

J. Zatorski, B. Sikora, S. G. Karshenboim, S. Sturm, F. Köhler-Langes, K. Blaum, C. H. Keitel, Z. Harman, Phys. Rev. A 96, 012502, 2017]

Uncertainty is dominated by the experiment but improvement by an order of magnitude is expected

Why do we measure the bound electron g-factor

- Currently, it is used to determine m_e
- Future plans
  - Determination of $^4He^+$ mass
  - New measurement of fine structure constant – g rather than $g-2$ is measured
    – a large reduction of relativistic shifts compared to free electron, nucleus acts as an anchor
    – combination of measurements for different energy levels allows canceling leading nuclear effects
  - Some proposals suggest measurement that will not depend on the free $g-2$ contribution
  - Test of QED in strong fields

Future prospects

Experiments designed to provide tests of bound state QED

- Mainz g-factor experiment
- ALPHATRAP (MPI-K Heidelberg)
- HITRAP (GSI Darmstadt)

New independent source of α!

- Combine Li-like H-like ions to cancel dependence on the finite nuclear size effects
- Use different nuclei to cancel dependence on the free electron $g-2$ [V.A. Yerokhin et al., Phys.Rev.Lett., 116, 100801, 2016]

or

- Use heavy ions $Z \gg 1$ and construct a function of g-factors for different energy levels such that the finite nuclear size effects cancel and α dependence is enhanced by Z
  [V.M. Shabaev et al., Phys.Rev.Lett., 96, 253002, 2006]

Bound muon g-factor

Recent preprint:

Access to improve the muon mass and magnetic moment anomaly via the bound-muon g factor

B. Sikora,[1,*] H. Cakir,[1] N. Michel,[1] V. Debierre,[1] N. S. Oreshkina,[1]
N. A. Belov,[1] V. A. Yerokhin,[1,2] C. H. Keitel,[1] and Z. Harman[1,†]

[1]*Max Planck Institute for Nuclear Physics, Saupfercheckweg 1, 69117 Heidelberg, Germany*
[2]*Center for Advanced Studies, Peter the Great St. Petersburg Polytechnic University, 195251 St. Petersburg, Russia*
(Dated: January 9, 2018)

- Theoretical accuracy 2×10^{-9} can be achieved for Helium. [Sikora, et al., arXiv:1801.02501]
- Bohr radius of muon is much smaller $\frac{r_{B\mu}}{r_{Be}} \sim \frac{m_e}{m_\mu}$ – certain contributions are enhanced
- Nuclear size effects and corrections induced by electron loops are highly enhanced!

Experimentally such precision will be challenging but it could provide an independent source of information about muon properties.

Effect	Term	Numerical value	Ref.
Dirac value		1.999 857 988 8	[19, 21]
Finite nuclear size		0.000 000 094 6(4)	[45]
One-loop SE	$(Z\alpha)^0$	0.002 322 819 5	[19, 46]
	all-order binding	0.000 000 084 9(10)	
One-loop VP	eVP, Uehling	-0.000 000 479 6	
	eVP, magnetic loop	0.000 000 127 2(4)	
	μVP, Uehling	-0.000 000 000 1	
	hadronic VP, Uehling	-0.000 000 000 1(1)	
Two-loop QED	$(Z\alpha)^0$	0.000 008 264 4	[47, 48]
	SE-SE, $(Z\alpha)^2$— $(Z\alpha)^5$	-0.000 000 000 1	[13, 49–51]
	S(eVP)E, $(Z\alpha)^2$	0.000 000 000 4	[47–50]
	2nd-order Uehling	-0.000 000 001 1(4)	
	Källén-Sabry	-0.000 000 003 5	
	magnetic loop+Uehling	0.000 000 000 3	
$\geq$ Three-loop QED	$(Z\alpha)^0$	0.000 000 610 6	[19, 52–54]
Nuclear recoil	$(\frac{m}{M})^1$, all orders in $Z\alpha$	0.000 006 038 2	[55]
	$(\frac{m}{M})^{2+}$, $(Z\alpha)^2$	-0.000 000 488 7	[56]
	radiative recoil	-0.000 000 004 7	[57]
Weak interaction	$(Z\alpha)^0$	0.000 000 003 1	[19, 58]
Hadronic contributions	$(Z\alpha)^0$	0.000 000 139 3(12)	[19, 59–61]
Sum		2.002 195 193 4(20)	

[Sikora, et al., arXiv:1801.02501]

Existing data

TABLE I. Measured frequencies and amplitudes of the spin precession and the g-factor shift (relative to the free muon) of the negative muon bound in various atoms. For the positive muon in copper the precession frequency is corrected for the value of the Knight shift [16].

Sample	Frequency (MHz)	P/3 (%)	$10^4 \times (g_\mu^{\text{free}} - g_\mu^{1s})/g_\mu^{\text{free}}$			
Cu (μ^+)	33.8772±0.0007					
C	33.8523±0.0013	5.30±0.03	7.3±0.4[a]	7.5±0.2[b]	7.18±0.23[c]	8.2[d]
Zn	33.652±0.021	1.73±0.05	67±6[a]	75±9[b]	115.0±2.6[c]	129[d]
Zn(0.9999)	33.625±0.021	1.68±0.05	73±6[a]			
Ge	33.617±0.022	1.69±0.04	77±7[a]			
Cd	33.734±0.036	1.10±0.04	42±10[a]	67±22[b]	215^{+17}_{-21} [c]	218[d]
Pb(0.99999)	33.625±0.084	1.60±0.10	75±25[a]		260^{+22}_{-23} [c]	383[d]

[a]Present results.
[b]Experimental data from Ref. [6].
[c]Experimental data from Ref. [8].
[d]Theoretical calculation from Ref. [3].

Table from [T. N. Mamedov, K. I. Gritsay, A. V. Stoykov, D. Herlach, R. Scheuermann, and U. Zimmermann, Phys. Rev. A 75, 054501, 2007]

Conclusions

- Spectroscopic measurements serve as the most precise source of fundamental constants and further progress is expected
- Bound electron g-factor can help to check muon g-2, thanks to significant progress in spectroscopy and theoretical computations
- Hopefully, in the future, the bound muon g-factor could be measured with high accuracy.