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DISPERSIVE APPROACH TO QCD AND ITS APPLICATIONS

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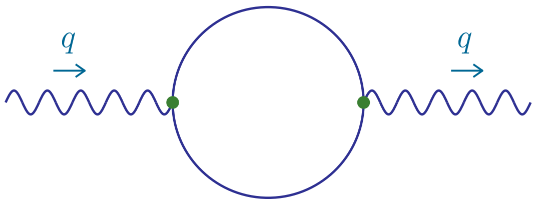
INTRODUCTION

Hadronic vacuum polarization function $\Pi(q^2)$ plays a central role in various issues of QCD and Standard Model. In particular, the theoretical description of some strong interaction processes and of hadronic contributions to electroweak observables is inherently based on $\Pi(q^2)$:

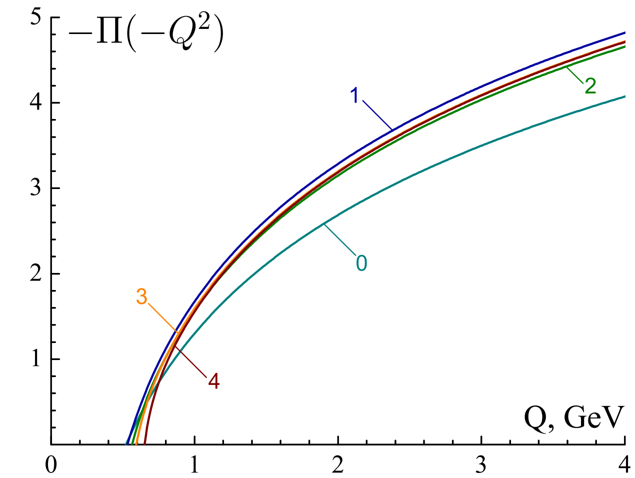
- **electron–positron annihilation into hadrons**
- **inclusive τ lepton hadronic decay**
- **muon anomalous magnetic moment**
- **running of the electromagnetic coupling**

QCD PERTURBATIVE PREDICTIONS

Leading order:



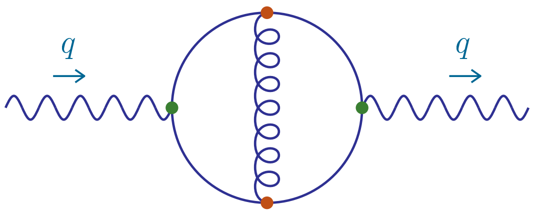
$$\Pi^{(0)}(q^2) = -\ln\left(\frac{-q^2}{-q_0^2}\right)$$

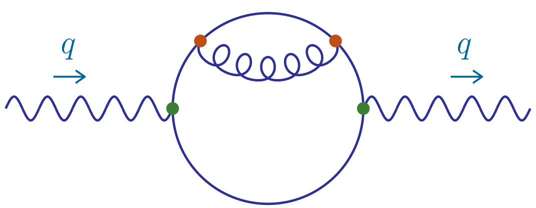


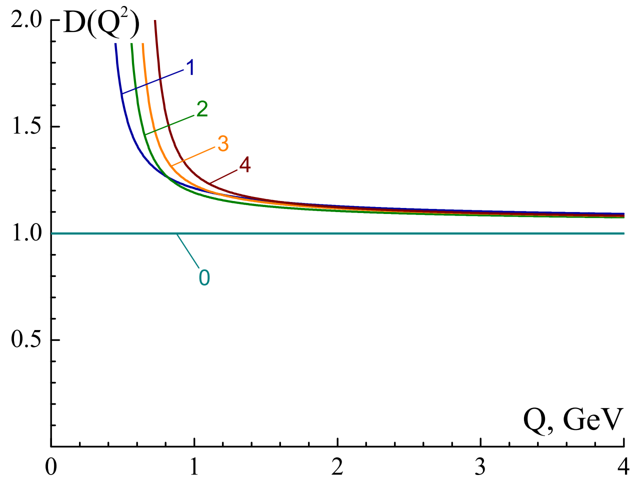
$$D^{(0)}(Q^2) = -\frac{d\,\Pi^{(0)}(-Q^2)}{d\ln Q^2} = 1, \qquad Q^2 = -q^2 > 0$$

[SL]

Strong corrections:







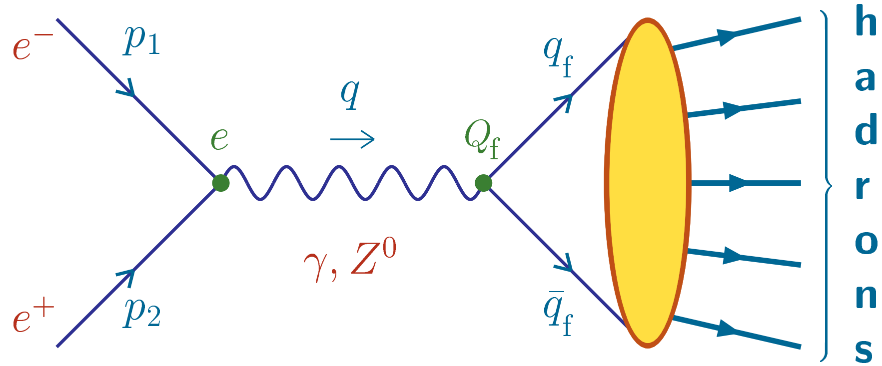
$$D^{(1)}(Q^2) = 1 + \frac{1}{\pi}\,\alpha_s^{(1)}(Q^2) = 1 + \frac{4}{\beta_0}\frac{1}{\ln(Q^2/\Lambda^2)}$$

$$D^{(\ell)}(Q^2) = 1 + \sum_{j=1}^{\ell} d_j \left[\alpha_s^{(\ell)}(Q^2)\beta_0/(4\pi)\right]^j = 1 + \sum_{j=1}^{\ell} d_j \left[a_s^{(\ell)}(Q^2)\right]^j$$

The factor $N_c \sum_{f=1}^{n_f} Q_f^2$ is omitted unless otherwise specified

GENERAL DISPERSION RELATIONS

Cross–section of $e^+e^- \to$ hadrons:



$$\sigma = 4\pi^2 \frac{2\alpha^2}{s^3} L^{\mu\nu} \Delta_{\mu\nu},$$

where $s = q^2 = (p_1 + p_2)^2 > 0$ **[TL],**

$$L_{\mu\nu} = \frac{1}{2}\Big[q_\mu q_\nu - g_{\mu\nu} q^2 - (p_1 - p_2)_\mu (p_1 - p_2)_\nu\Big],$$

$$\Delta_{\mu\nu} = (2\pi)^4 \sum_\Gamma \delta(p_1 + p_2 - p_\Gamma) \langle 0 | J_\mu(-q) | \Gamma \rangle \langle \Gamma | J_\nu(q) | 0 \rangle,$$

and $J_\mu = \sum_f Q_f : \bar{q}\, \gamma_\mu\, q :$ **is the electromagnetic quark current.**

Kinematic restriction: **the hadronic tensor** $\Delta_{\mu\nu}(q^2)$ **assumes non–zero values only for** $q^2 \geq 4m_\pi^2 = m^2$**, since otherwise no hadron state** Γ **could be excited**

■ **Feynman (1972); Adler (1974)**

The hadronic tensor can be represented as $\Delta_{\mu\nu} = 2\,\mathrm{Im}\,\Pi_{\mu\nu}$,

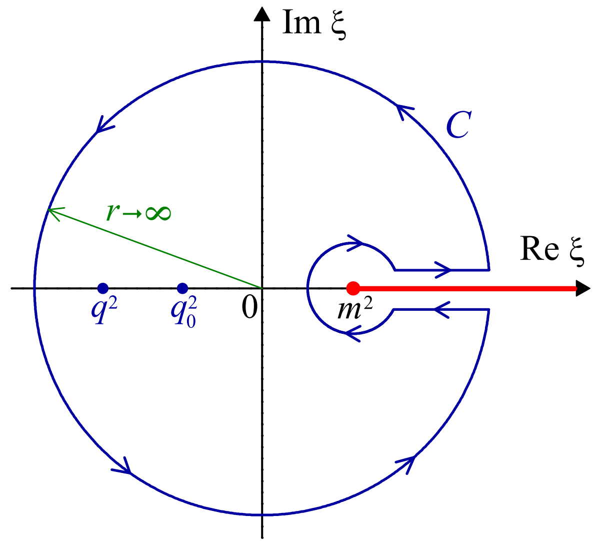
$$\Pi_{\mu\nu}(q^2) = i\int e^{iqx}\langle 0|\, T\{J_\mu(x)\, J_\nu(0)\}\,|0\rangle\, d^4x = i\big(q_\mu q_\nu - g_{\mu\nu}q^2\big)\frac{\Pi(q^2)}{12\pi^2}.$$

Kinematic restriction: $\Pi(q^2)$ **has the only cut** $s = q^2 \geq m^2$

Dispersion relation for $\Pi(q^2)$**:**

$$\Delta\Pi(q^2,\, q_0^2) = \frac{1}{2\pi i}(q^2 - q_0^2)\oint_C \frac{\Pi(\xi)}{(\xi - q^2)(\xi - q_0^2)}\,d\xi$$

$$= (q^2 - q_0^2)\int_{m^2}^{\infty}\frac{R(s)}{(s - q^2)(s - q_0^2)}\,ds,$$



where $\Delta\Pi(q^2,\, q_0^2) = \Pi(q^2) - \Pi(q_0^2)$ **and** $R(s)$ **denotes the measurable ratio of two cross–sections**

$$R(s) = \frac{1}{2\pi i}\lim_{\varepsilon\to 0_+}\Big[\Pi(s + i\varepsilon) - \Pi(s - i\varepsilon)\Big] = \frac{\sigma(e^+e^- \to \text{hadrons};\, s)}{\sigma(e^+e^- \to \mu^+\mu^-;\, s)}$$

Kinematic restriction: $R(s) = 0$ **for** $s = q^2 < m^2$

In general, it is also convenient to employ the so–called Adler function ($Q^2 = -q^2 > 0$ [SL])

$$D(Q^2) = -\frac{d\,\Pi(-Q^2)}{d\ln Q^2}, \qquad D(Q^2) = Q^2 \int_{m^2}^{\infty} \frac{R(s)}{(s+Q^2)^2}\, ds$$

■ Adler (1974); De Rujula, Georgi (1976); Bjorken (1989)

This dispersion relation provides a link between experimentally measurable and theoretically computable quantities.

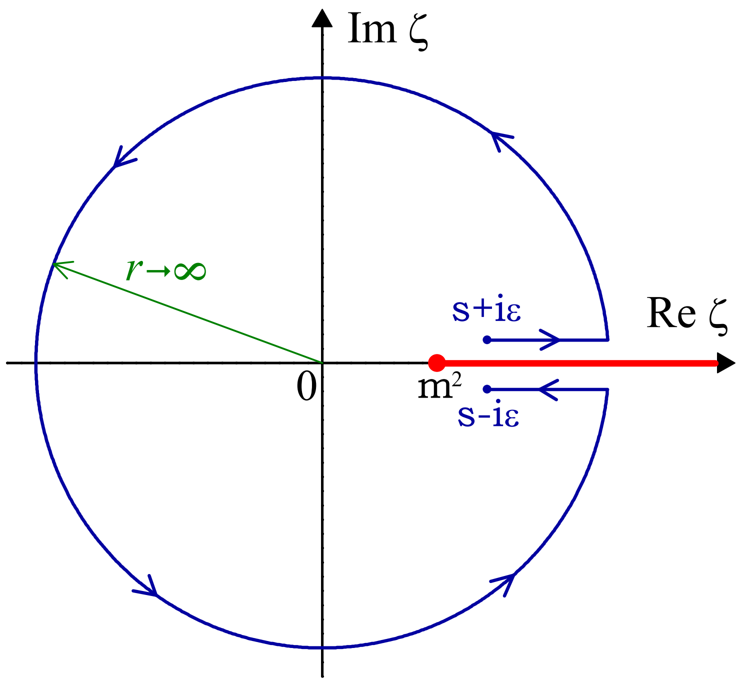
The inverse relations between the functions on hand read

$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \int_{s+i\varepsilon}^{s-i\varepsilon} D(-\zeta)\, \frac{d\zeta}{\zeta},$$

■ Radyushkin (1982); Krasnikov, Pivovarov (1982)

$$\Delta\Pi(-Q^2, -Q_0^2) = -\int_{Q_0^2}^{Q^2} D(\sigma)\, \frac{d\sigma}{\sigma}$$

■ Pivovarov (1992)



The complete set of relations between $\Pi(q^2)$, $R(s)$, and $D(Q^2)$:

$$\Delta\Pi(q^2,\, q_0^2) = (q^2 - q_0^2) \int_{m^2}^{\infty} \frac{R(\sigma)}{(\sigma - q^2)(\sigma - q_0^2)}\, d\sigma = - \int_{-q_0^2}^{-q^2} D(\sigma)\, \frac{d\sigma}{\sigma},$$

$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \Big[\Pi(s + i\varepsilon) - \Pi(s - i\varepsilon)\Big] = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \int_{s+i\varepsilon}^{s-i\varepsilon} D(-\zeta)\frac{d\zeta}{\zeta},$$

$$D(Q^2) = -\frac{d\,\Pi(-Q^2)}{d\,\ln Q^2} = Q^2 \int_{m^2}^{\infty} \frac{R(\sigma)}{(\sigma + Q^2)^2}\, d\sigma.$$

Derivation of these relations requires only the location of cut of $\Pi(q^2)$ and its UV asymptotic. Neither additional approximations nor phenomenological assumptions are involved.

Nonperturbative constraints:

- $\Pi(q^2)$: **has the only cut** $q^2 \geq m^2$
- $R(s)$: **embodies "SL ⟶ TL" effects, vanishes for** $s < m^2$
- $D(Q^2)$: **has the only cut** $Q^2 \leq -m^2$, **vanishes at** $Q^2 \to 0$

DISPERSIVE APPROACH TO QCD

Functions on hand in terms of the common spectral density:

$$\Delta\Pi(q^2,\, q_0^2) = \Delta\Pi^{(0)}(q^2,\, q_0^2) + \int_{m^2}^{\infty} \rho(\sigma) \ln\left(\frac{\sigma - q^2}{\sigma - q_0^2}\,\frac{m^2 - q_0^2}{m^2 - q^2}\right)\frac{d\,\sigma}{\sigma},$$

$$R(s) = R^{(0)}(s) + \theta(s - m^2) \int_{s}^{\infty} \rho(\sigma)\,\frac{d\,\sigma}{\sigma},$$

$$D(Q^2) = D^{(0)}(Q^2) + \frac{Q^2}{Q^2 + m^2} \int_{m^2}^{\infty} \rho(\sigma)\,\frac{\sigma - m^2}{\sigma + Q^2}\,\frac{d\,\sigma}{\sigma},$$

$$\rho(\sigma) = \frac{1}{\pi}\frac{d}{d\,\ln\sigma}\,\mathrm{Im}\lim_{\varepsilon\to 0_+} p(\sigma - i\varepsilon) = -\frac{d\,r(\sigma)}{d\,\ln\sigma} = \frac{1}{\pi}\,\mathrm{Im}\lim_{\varepsilon\to 0_+} d(-\sigma - i\varepsilon),$$

where $\Delta\Pi^{(0)}(q^2,\, q_0^2)$, $R^{(0)}(s)$, $D^{(0)}(Q^2)$ denote the leading–order terms and $p(q^2)$, $r(s)$, $d(Q^2)$ stand for the strong corrections

■ **Nesterenko, Papavassiliou (2005–2007); Nesterenko (2007–2014)**

Derivation of obtained representations involves neither additional approximations nor model–dependent assumptions, with all the nonperturbative constraints being embodied.

The leading–order terms of the functions on hand read

$$\Delta\Pi^{(0)}(q^2,\, q_0^2) = 2\,\frac{\varphi - \tan\varphi}{\tan^3\varphi} - 2\,\frac{\varphi_0 - \tan\varphi_0}{\tan^3\varphi_0}, \qquad \sin^2\varphi = \frac{q^2}{m^2},$$

$$R^{(0)}(s) = \theta(s-m^2)\left(1 - \frac{m^2}{s}\right)^{3/2}, \qquad \sin^2\varphi_0 = \frac{q_0^2}{m^2},$$

$$D^{(0)}(Q^2) = 1 + \frac{3}{\xi}\left[1 - \sqrt{1+\xi^{-1}}\,\sinh^{-1}\left(\xi^{1/2}\right)\right], \qquad \xi = \frac{Q^2}{m^2}$$

■ Feynman (1972); Akhiezer, Berestetsky (1965)

Perturbative contribution to the spectral density:

$$\rho_{\rm pert}(\sigma) = \frac{1}{\pi}\frac{d\,\mathrm{Im}\,p_{\rm pert}(\sigma - i0_+)}{d\,\ln\sigma} = -\frac{d\,r_{\rm pert}(\sigma)}{d\,\ln\sigma} = \frac{1}{\pi}\,\mathrm{Im}\,d_{\rm pert}(-\sigma - i0_+)$$

one–loop: $\rho^{(1)}_{\rm pert}(\sigma) = 4/\left[\beta_0\left(\ln^2(\sigma/\Lambda^2) + \pi^2\right)\right]$**; early attempts for the higher–loops:** ■ Nesterenko, Simolo (2010, 2011); Bakulev (2013); Cvetic (2015)

The perturbative spectral function at the ℓ–loop level:

$$\rho^{(\ell)}_{\text{pert}}(\sigma) = \sum_{j=1}^{\ell} d_j\, \bar{\rho}^{(\ell)}_j(\sigma), \qquad \bar{\rho}^{(\ell)}_j(\sigma) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \left\{ \left[a^{(\ell)}_{\text{s}}(-\sigma - i\varepsilon) \right]^j - \left[a^{(\ell)}_{\text{s}}(-\sigma + i\varepsilon) \right]^j \right\}.$$

Explicit expression for $\rho^{(\ell)}_{\text{pert}}(\sigma)$ valid at an arbitrary loop level:

$$\rho^{(\ell)}_{\text{pert}}(\sigma) = \sum_{j=1}^{\ell} d_j \sum_{k=0}^{K(j)} \binom{j}{2k+1} (-1)^k \pi^{2k} \left[\sum_{n=1}^{\ell} \sum_{m=0}^{n-1} b^m_n\, u^m_n(\sigma) \right]^{j-2k-1} \left[\sum_{n=1}^{\ell} \sum_{m=0}^{n-1} b^m_n\, v^m_n(\sigma) \right]^{2k+1}$$

■ **Nesterenko (2016, 2017)**

In this equation ℓ denotes the loop level,

$$u^m_n(\sigma) = \begin{cases} u^0_n(\sigma), & \text{if } m = 0, \\ u^0_n(\sigma) u^m_0(\sigma) - \pi^2 v^0_n(\sigma) v^m_0(\sigma), & \text{if } m \geq 1, \end{cases}$$

$$v^m_n(\sigma) = \begin{cases} v^0_n(\sigma), & \text{if } m = 0, \\ v^0_n(\sigma) u^m_0(\sigma) + u^0_n(\sigma) v^m_0(\sigma), & \text{if } m \geq 1, \end{cases}$$

$$v_0^m(\sigma) = \sum_{k=0}^{K(m)} \binom{m}{2k+1} (-1)^{k+1} \pi^{2k} \Big[L_1(y)\Big]^{m-2k-1} \Big[L_2(y)\Big]^{2k+1},$$

$$u_0^m(\sigma) = \sum_{k=0}^{K(m+1)} \binom{m}{2k} (-1)^{k} \pi^{2k} \Big[L_1(y)\Big]^{m-2k} \Big[L_2(y)\Big]^{2k},$$

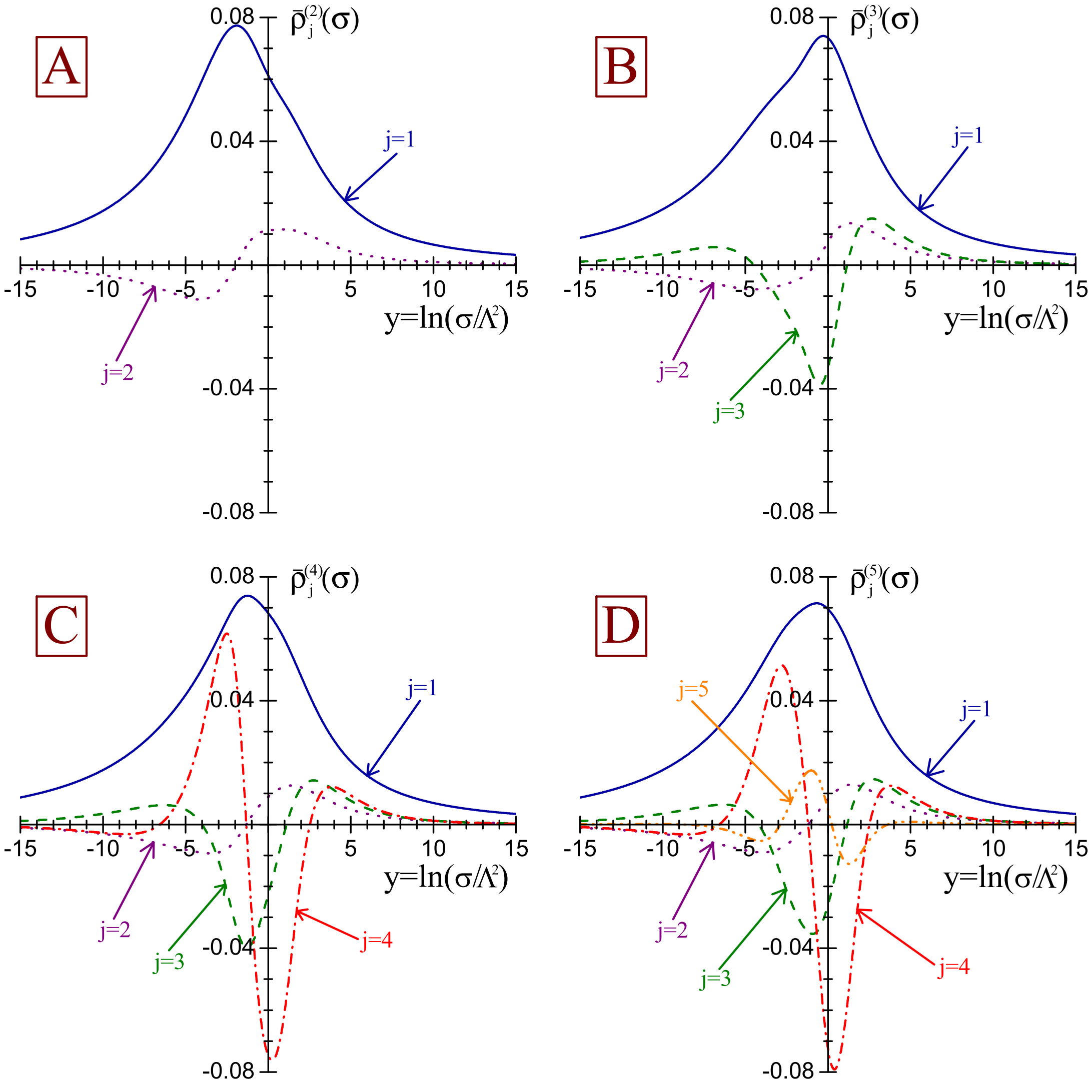
$$v_n^0(\sigma) = \frac{1}{(y^2+\pi^2)^n} \sum_{k=0}^{K(n)} \binom{n}{2k+1} (-1)^{k} \pi^{2k} y^{n-2k-1}, \qquad L_1(y) = \ln\sqrt{y^2+\pi^2},$$

$$u_n^0(\sigma) = \frac{1}{(y^2+\pi^2)^n} \sum_{k=0}^{K(n+1)} \binom{n}{2k} (-1)^{k} \pi^{2k} y^{n-2k}, \qquad L_2(y) = \frac{1}{2} - \frac{1}{\pi}\arctan\left(\frac{y}{\pi}\right),$$

$$K(n) = \frac{n-2}{2} + \frac{n \bmod 2}{2}, \qquad \binom{n}{m} = \frac{n!}{m!\,(n-m)!}, \qquad y = \ln\left(\frac{\sigma}{\Lambda^2}\right),$$

and b_n^m stands for a combination of the β function perturbative expansion coefficients ($b_1^0 = 1$, $b_2^0 = 0$, $b_2^1 = -\beta_1/\beta_0^2$, etc.)

■ **Nesterenko (2016, 2017)**



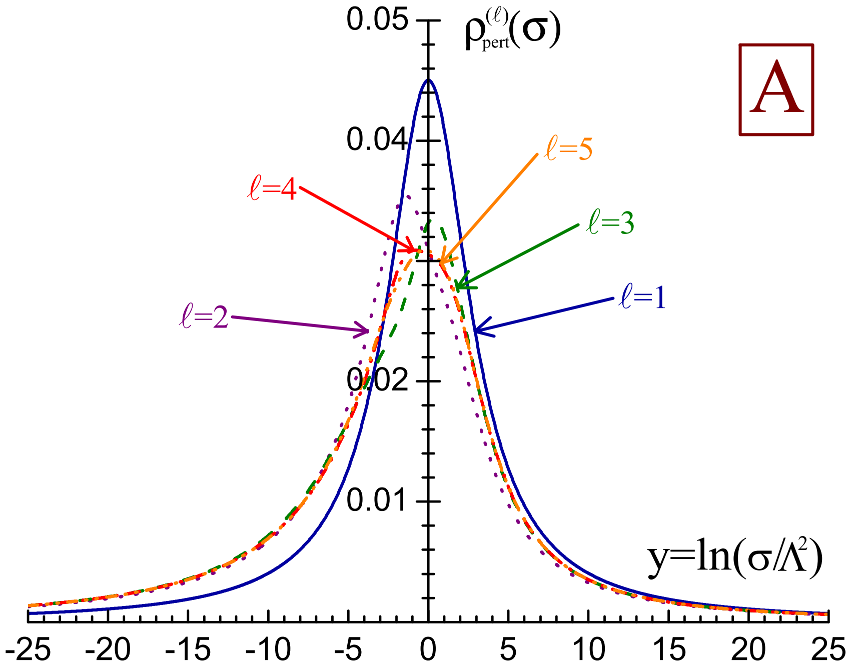
Functions $\bar{\rho}_j^{(\ell)}(\sigma)$ are scaled by 10, 10^2, 10^2 for $j = 3, 4, 5$

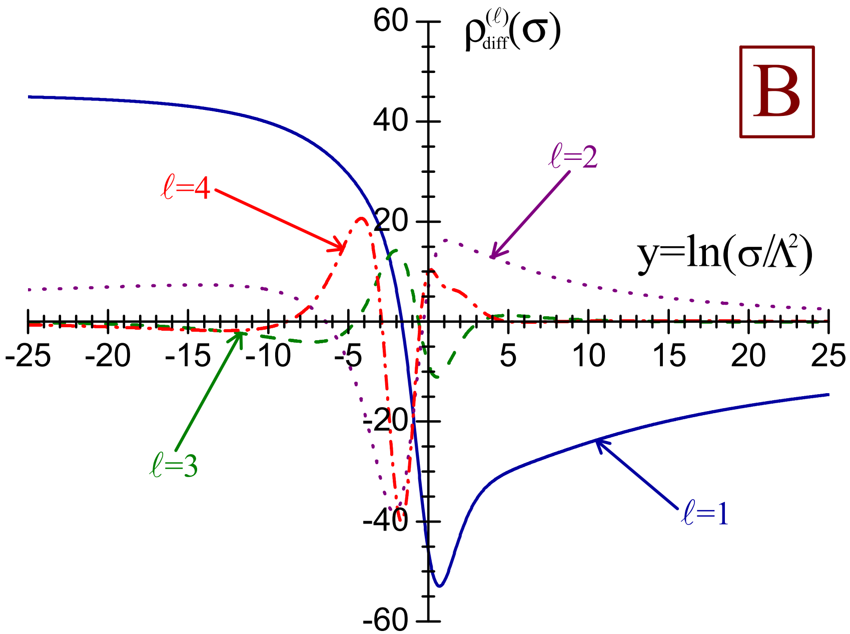
The higher–order partial spectral functions $\bar{\rho}_j^{(\ell)}(\sigma)$ are suppressed with respect to those of the preceding orders. This subdominance eventually leads to an enhanced higher–loop and scheme stability of the outcoming results at moderate and low energies.

■ Nesterenko (2016, 2017)

Employed higher–loop perturbative coefficients:

- Baikov, Chetyrkin, Kuhn (2017); Herzog, Ruijl, Ueda, Vermaseren, Vogt (2017); Baikov, Chetyrkin, Kuhn, Rittinger (2012); Kataev, Starshenko (1995)





Function $\rho_{\text{diff}}^{(4)}(\sigma)$ is scaled by the factor of 10

The perturbative spectral function $\rho_{\text{pert}}^{(\ell)}(\sigma)$ is remarkably stable with respect to the higher–loop corrections. In particular, the range of y, where the difference between $\rho_{\text{pert}}^{(\ell)}(\sigma)$ and $\rho_{\text{pert}}^{(\ell+1)}(\sigma)$ is sizable, is located in the vicinity of $y = 0$ and becomes smaller at larger ℓ.

Plot A: $\rho_{\text{pert}}^{(\ell)}(\sigma)$ for $\ell = 1 \ldots 5$

Plot B: $\rho_{\text{diff}}^{(\ell)}(\sigma)$ for $\ell = 1 \ldots 4$

$$\rho_{\text{diff}}^{(\ell)}(\sigma) = \left[1 - \frac{\rho_{\text{pert}}^{(\ell)}(\sigma)}{\rho_{\text{pert}}^{(\ell+1)}(\sigma)}\right] \times 100\%$$

■ **Nesterenko (2016, 2017)**

Note on the massless limit

In the limit $m = 0$ the obtained integral representations read

$$\Delta\Pi(q^2,\, q_0^2) = -\ln\left(\frac{-q^2}{-q_0^2}\right) + \int_0^\infty \rho(\sigma) \ln\left[\frac{1-(\sigma/q^2)}{1-(\sigma/q_0^2)}\right] \frac{d\,\sigma}{\sigma},$$

$$R(s) = 1 + \int_s^\infty \rho(\sigma)\, \frac{d\,\sigma}{\sigma}, \qquad D(Q^2) = 1 + \int_0^\infty \frac{\rho(\sigma)}{\sigma + Q^2}\, d\,\sigma.$$

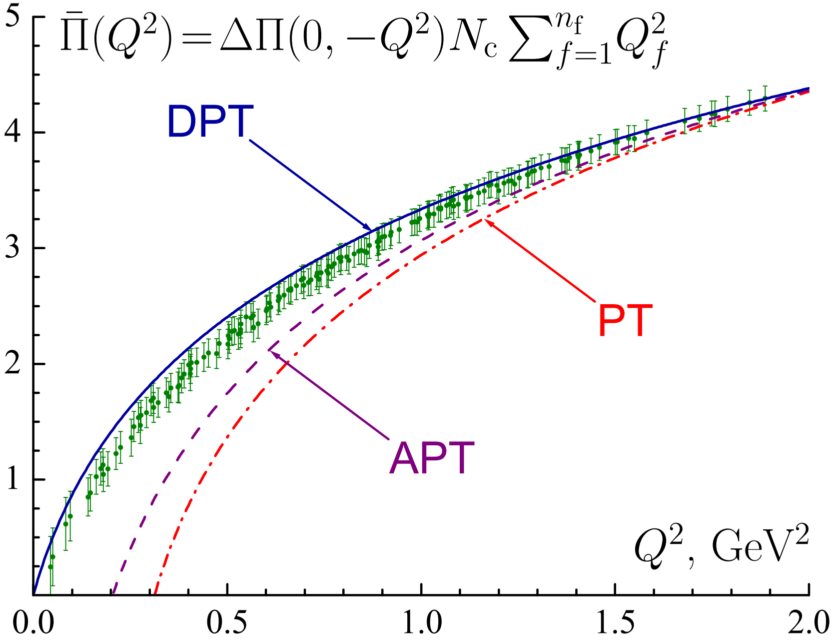
For $\rho(\sigma) = \rho_{\text{pert}}(\sigma)$ two highlighted massless equations become identical to those of the APT ■ Shirkov, Solovtsov, Milton (1997–2007)

$\Pi(q^2)$ was not addressed in the framework of the latter

However, it is essential to keep the threshold m nonvanishing:

- **massless limit loses some of nonperturbative constraints**
- **effects due to $m \neq 0$ become substantial at low energies**

Comparison of obtained results with lattice simulation data



Both PT and APT fail to describe $\Pi(q^2)$ at low energies:

PT: $\Pi(q^2)$ **possesses infrared unphysical singularities**

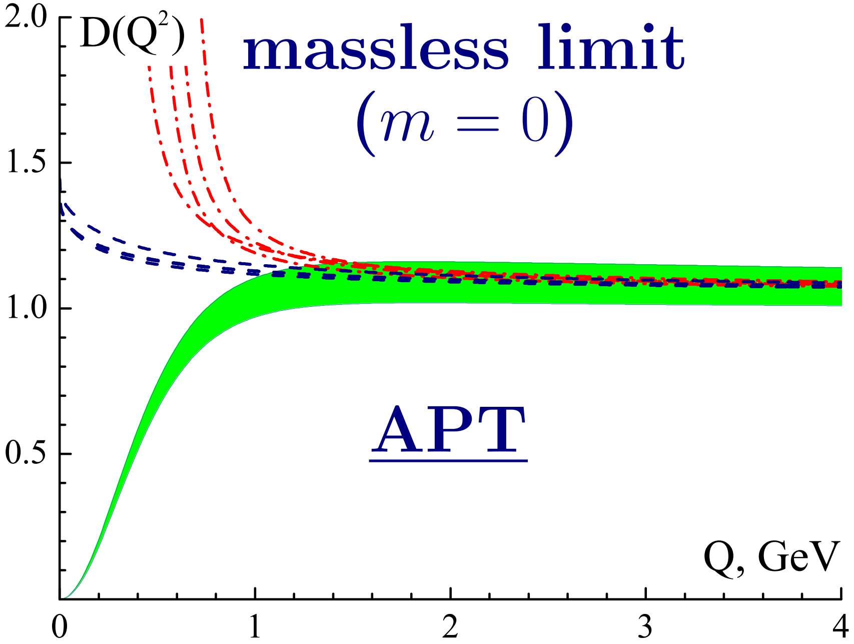
APT: $\Pi(q^2)$ **diverges in IR limit**

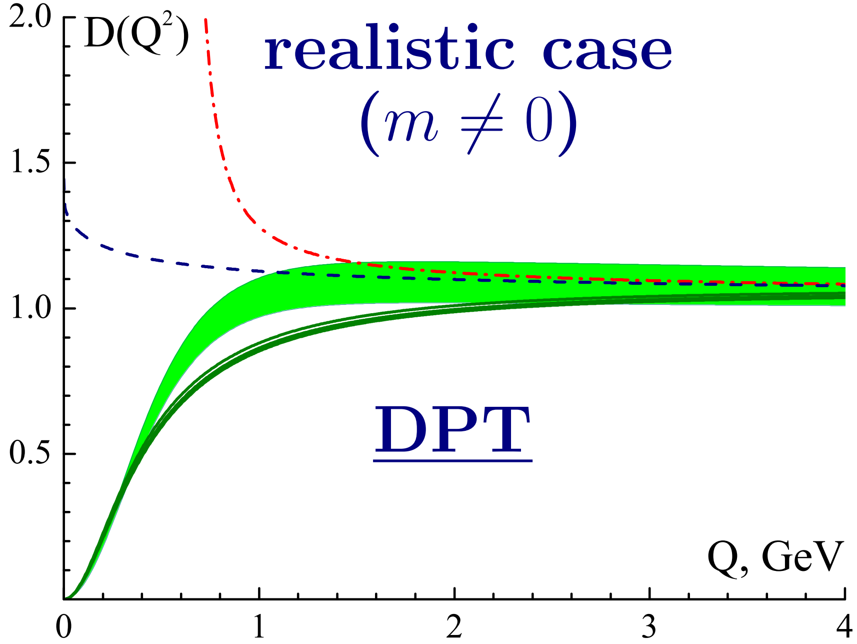
■ Della Morte, Jager, Juttner, Wittig (2011–2015); Nesterenko (2014, 2015)

	unphysical singularities	agreement with lattice
PT	contains	disagrees
APT	diverges in IR	disagrees
DPT	free	agrees

ADLER FUNCTION

Comparison of obtained results with experimental data



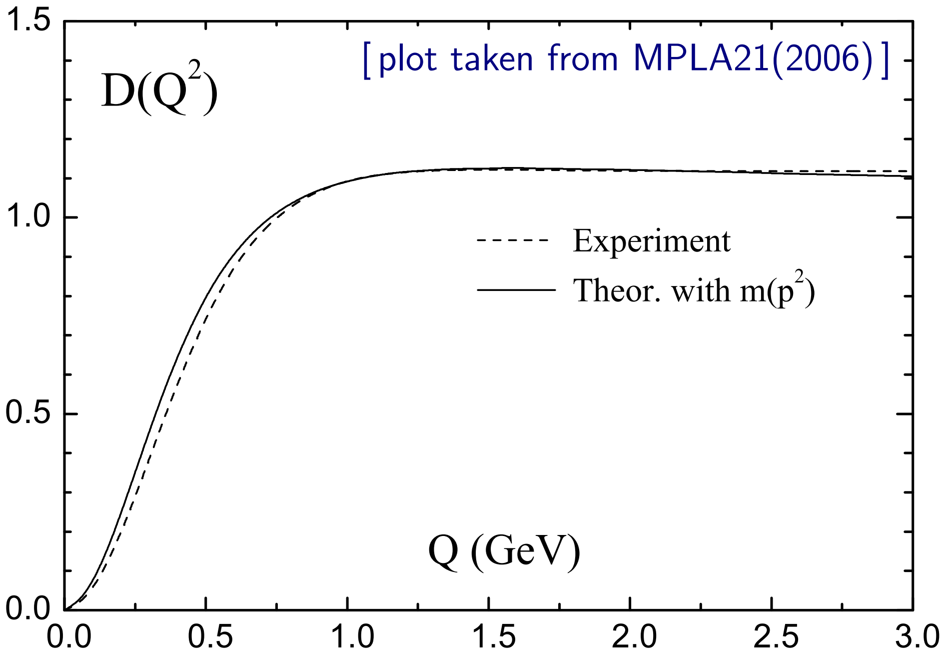


■ Nesterenko, Papavassiliou (2006); Nesterenko (2015, 2016)

	unphysical singularities	agreement with data
PT	contains	disagrees
APT	free	disagrees
DPT	free	agrees

Some attempts to improve IR behavior of $D(Q^2)$ within APT:

APT + relativistic quark mass threshold resummation:

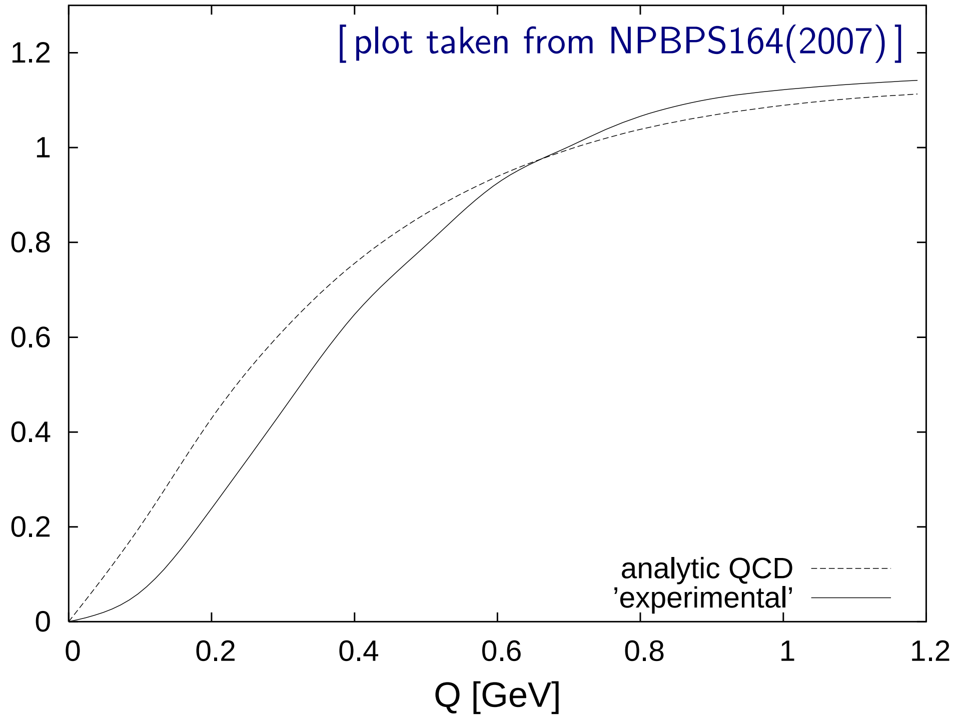


quite large light quark masses

$2m_{u,d} \simeq 520\,\text{MeV} \simeq 4m_\pi$

■ **Milton, Solovtsov, Solovtsova (2001–2006)**

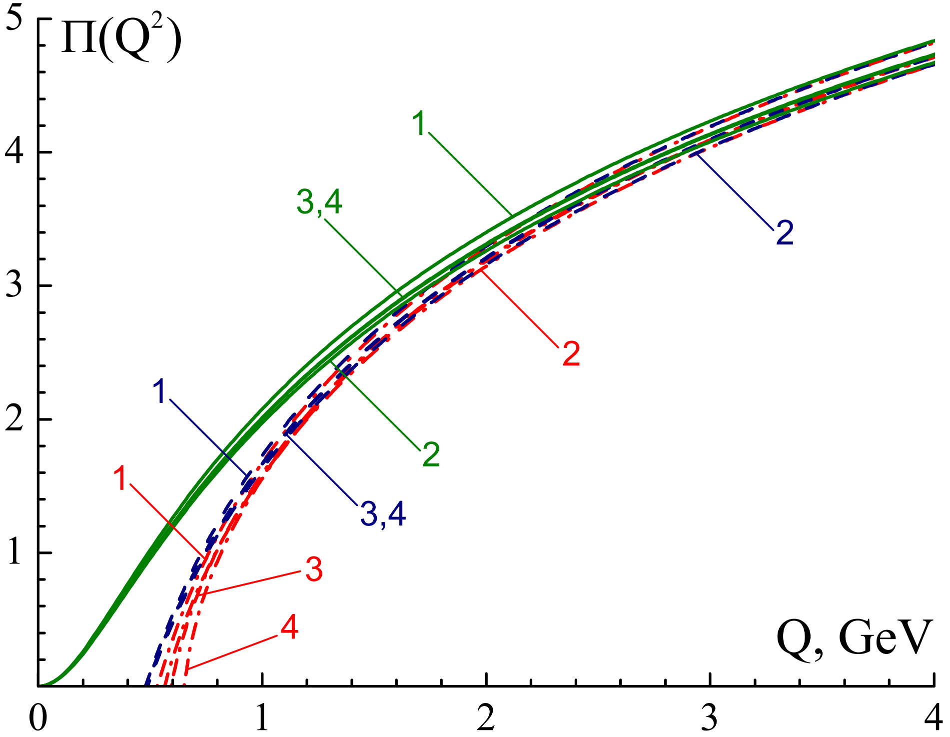
APT + vector meson dominance assumption:



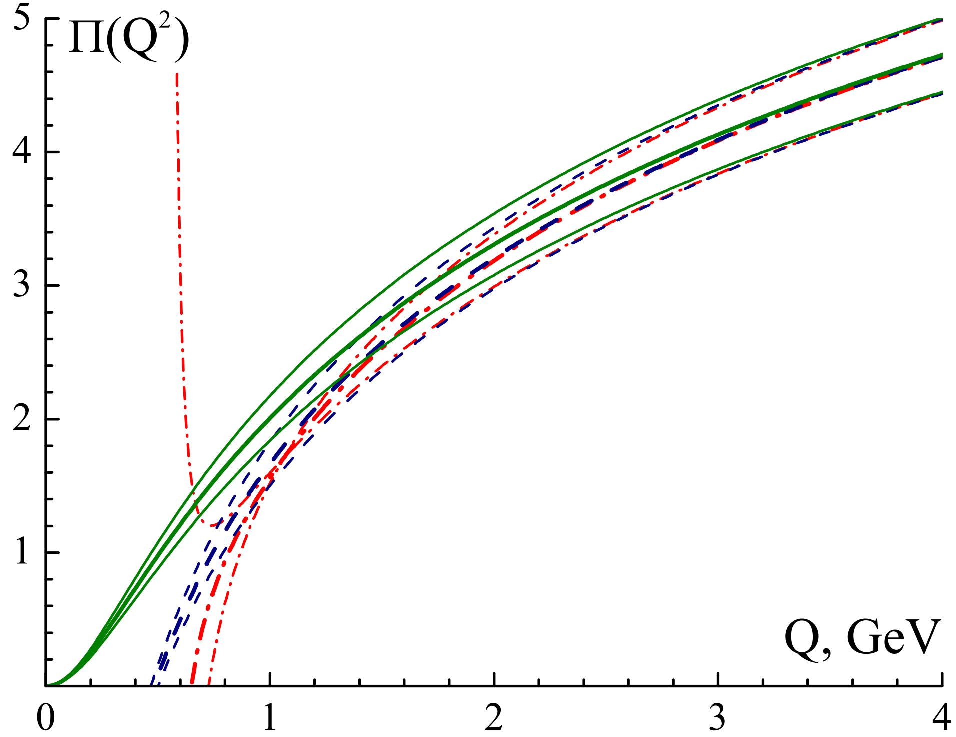
VMD NW approximation and cut–off at $M_0 \simeq 740\,\text{MeV}$

■ **Cvetic et al. (2005–2017)**

• Function $\Pi(q^2)$: higher loop and scheme stability



Loop levels: $\ell = 1 \ldots 4$



4–loop, "MS–like" scheme

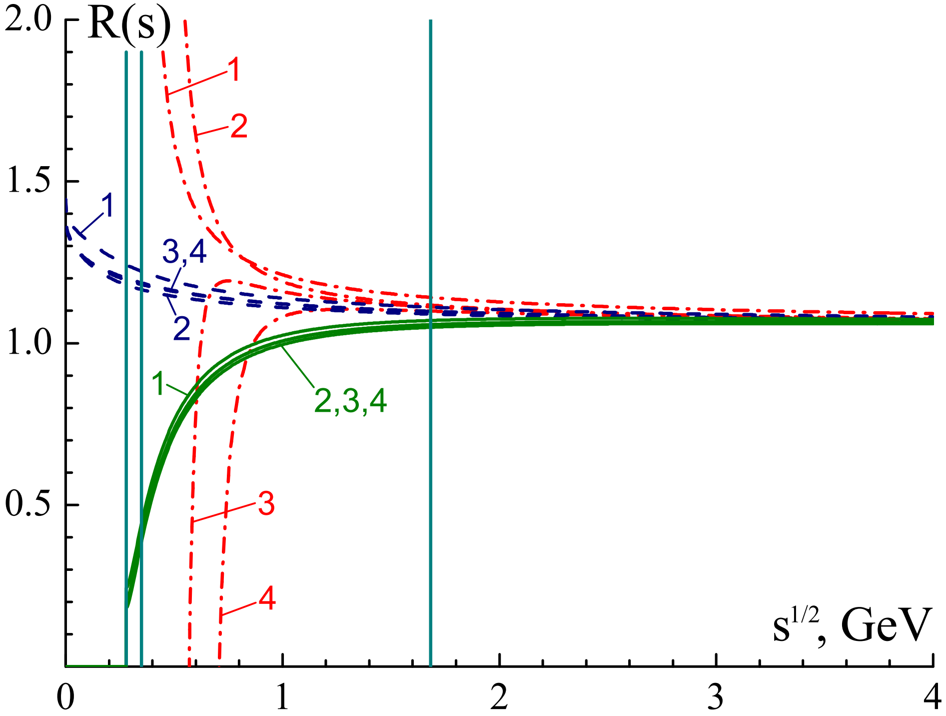
PT: dot–dashed curves

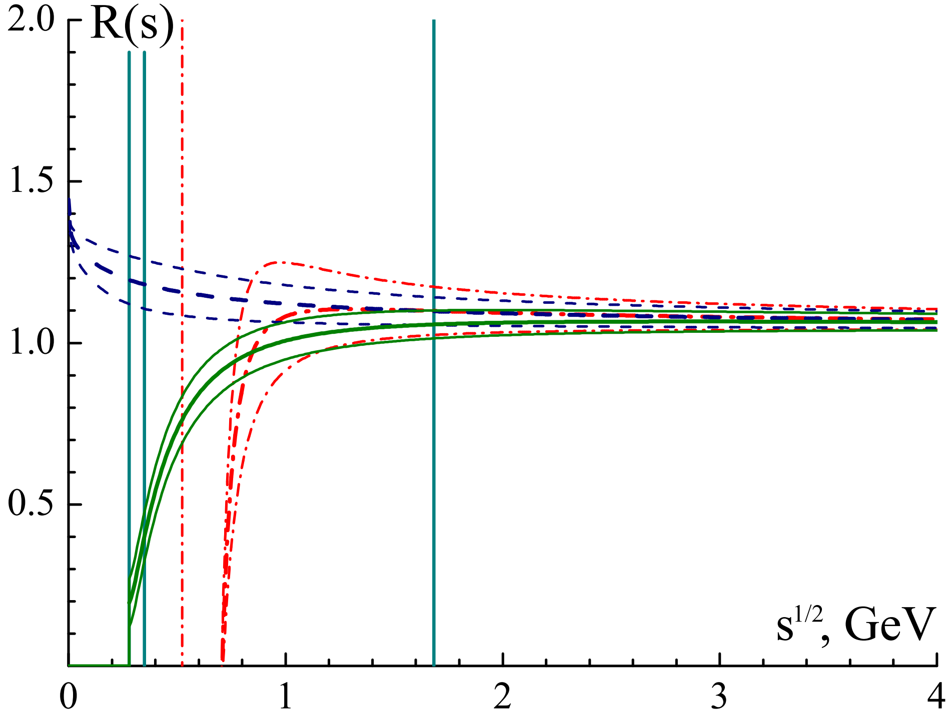
APT: dashed curves

DPT: solid curves

■ **Nesterenko (2016)**

• Function $R(s)$: higher loop and scheme stability





Loop levels: $\ell = 1 \ldots 4$ — **4–loop, "MS–like" scheme**

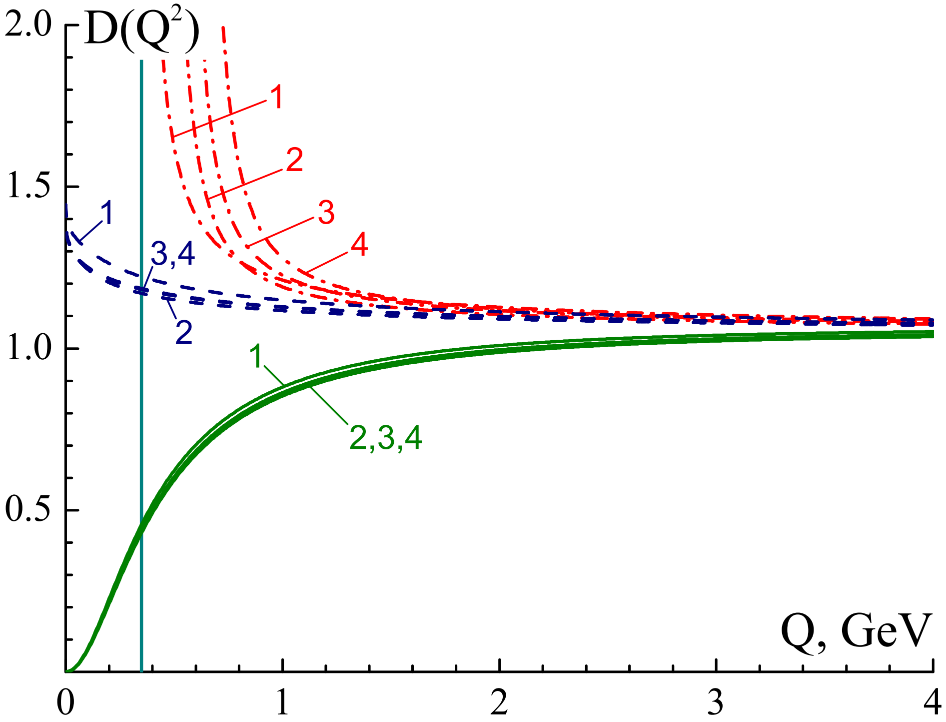
PT: dot–dashed curves

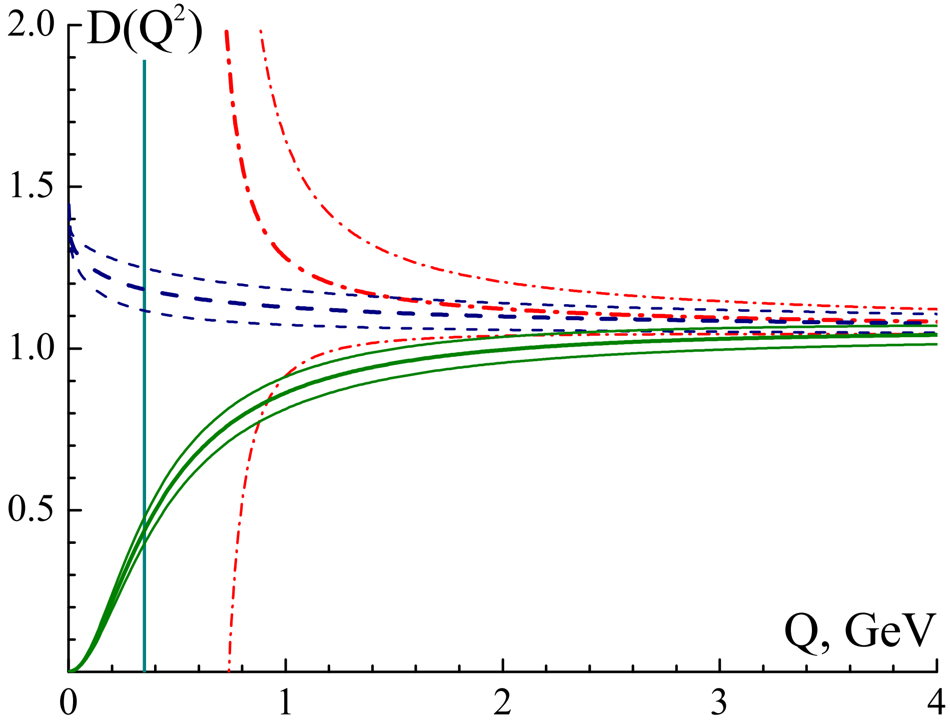
APT: dashed curves

DPT: solid curves

■ **Nesterenko (2016)**

• Function $D(Q^2)$: higher loop and scheme stability





Loop levels: $\ell = 1 \ldots 4$ **4–loop, "MS–like" scheme**

PT: dot–dashed curves

APT: dashed curves

DPT: solid curves

■ **Nesterenko (2016)**

MUON ANOMALOUS MAGNETIC MOMENT

The theoretical description of $a_\mu = (g_\mu - 2)/2$ is a long–standing challenging issue of the elementary particle physics.

Experiment: $a_\mu^{\rm exp} = (11659208.9 \pm 6.3) \times 10^{-10}$ (0.54 ppm)

- Muon (g-2) Collaboration (2006); Roberts (2010)

Theory: $a_\mu^{\rm theor} = a_\mu^{\rm QED} + a_\mu^{\rm EW} + a_\mu^{\rm HLO} + a_\mu^{\rm HHO} + a_\mu^{\rm Hlbl}$

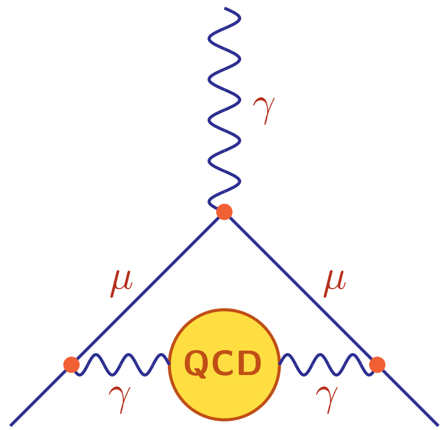
$a_\mu^{\rm QED} = (11658471.8951 \pm 0.0080) \times 10^{-10}$ Aoyama, Hayakawa, Kinoshita, Nio (2012)

$a_\mu^{\rm EW} = (15.36 \pm 0.10) \times 10^{-10}$ Gnendiger, Stockinger, Stockinger–Kim (2013)

$a_\mu^{\rm HHO} = (-9.84 \pm 0.07) \times 10^{-10}$ Hagiwara, Liao, Martin, Nomura, Teubner (2011)

$a_\mu^{\rm Hlbl} = (11.6 \pm 4.0) \times 10^{-10}$ Nyffeler (2014)

The uncertainty of theoretical estimation of a_μ is mainly dominated by the leading–order hadronic contribution $a_\mu^{\rm HLO}$ →



The latter involves the integration of $\Pi(q^2)$ over low energies:

$$a_\mu^{\text{HLO}} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty f\left(\frac{\zeta}{4m_\mu^2}\right)\bar{\Pi}(\zeta)\,\frac{d\zeta}{4m_\mu^2}, \qquad f(x) = \frac{1}{x^3}\,\frac{y^5(x)}{1-y(x)},$$

where $y(x) = x\left(\sqrt{1+x^{-1}} - 1\right)$ ■ Lautrup, Peterman, de Rafael (1972)

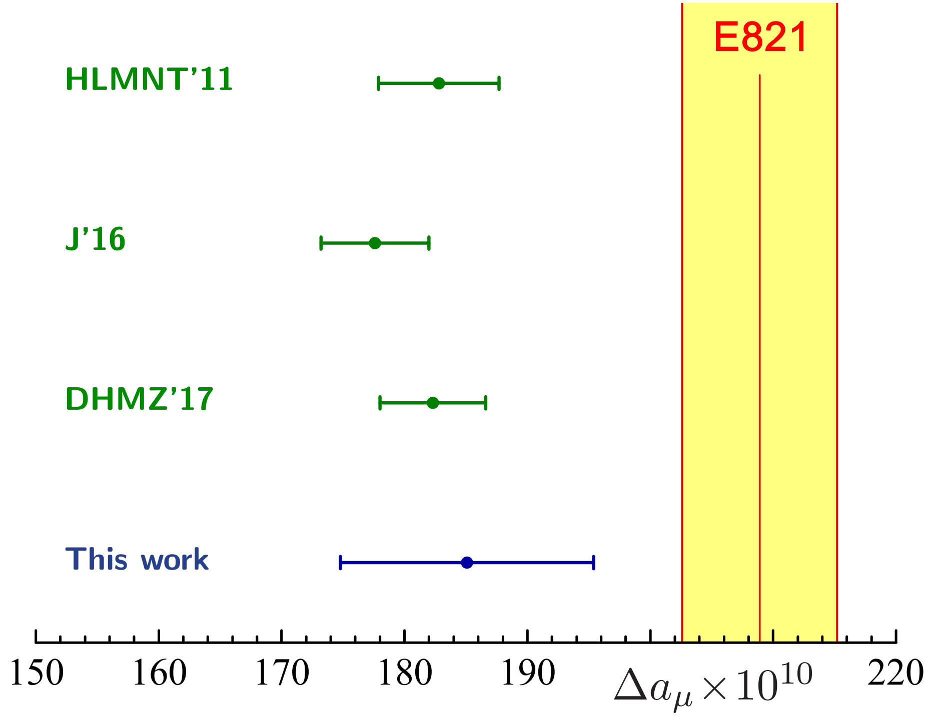
Dispersive approach enables one to evaluate a_μ^{HLO} without invoking experimental data on $R(s)$:

$$a_\mu^{\text{HLO}} = (696.1 \pm 9.5) \times 10^{-10}.$$

This result agrees fairly well with recent assessments of a_μ^{HLO}.

The complete SM prediction

$$a_\mu = (11659185.1 \pm 10.3) \times 10^{-10}$$



$\left[\,\Delta a_\mu = a_\mu - a_0, \quad a_0 = 11659 \times 10^{-7}\,\right]$

differs from a_μ^{exp} by two standard deviations ■ Nesterenko (2015)

The electromagnetic running coupling $\alpha_{\text{em}}(q^2)$ plays a central role in a variety of issues of precision particle physics:

$$\alpha_{\text{em}}(q^2) = \frac{\alpha}{1 - \Delta\alpha_{\text{lep}}(q^2) - \Delta\alpha_{\text{had}}(q^2)}$$

with $\alpha = e^2/(4\pi) \simeq 1/137.036$ being the fine structure constant.

Leptonic contribution to $\alpha_{\text{em}}(q^2)$ can be calculated within perturbation theory: $\Delta\alpha_{\text{lep}}(M_{\text{Z}}^2) = (314.979 \pm 0.002) \times 10^{-4}$ ■ Sturm (2013)

However, the respective hadronic contribution involves the integration over the low–energy range

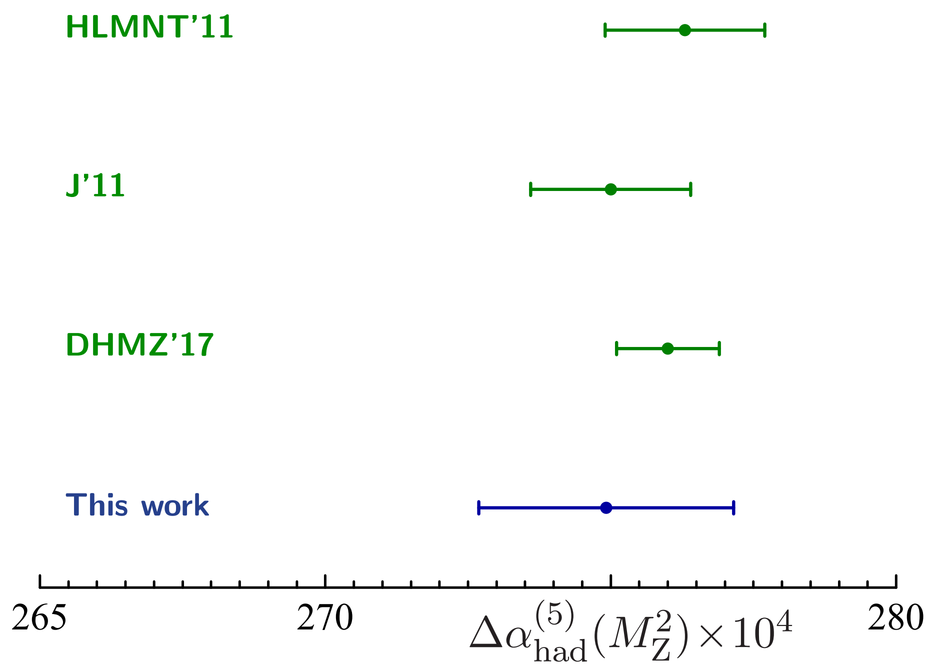
$$\Delta\alpha_{\text{had}}(M_{\text{Z}}^2) = -\frac{\alpha}{3\pi} M_{\text{Z}}^2 \fint_{m^2}^{\infty} \frac{R(s)}{s - M_{\text{Z}}^2} \frac{d\,s}{s}$$

and constitutes the prevalent source of uncertainty of $\alpha_{\text{em}}(M_{\text{Z}}^2)$.

As usual, the top quark contribution to $\alpha_{\text{em}}(q^2)$ is taken into account separately:

$$\Delta\alpha^{\text{top}}_{\text{had}}(M_Z^2) = (-0.70 \pm 0.05)\times 10^{-4}$$

■ Kuhn, Steinhauser (1998)



The evaluation of $\Delta\alpha^{(5)}_{\text{had}}(M_Z^2)$ in the framework of dispersive approach leads to

$$\Delta\alpha^{(5)}_{\text{had}}(M_Z^2) = (274.9 \pm 2.2)\times 10^{-4}.$$

The obtained assessment appears to be in a good agreement with recent estimations of $\Delta\alpha^{(5)}_{\text{had}}(M_Z^2)$ and eventually yields

$$\alpha^{-1}_{\text{em}}(M_Z^2) = 128.962 \pm 0.030$$

■ Nesterenko (2015)

PERTURBATIVE APPROXIMATION OF R–RATIO

The only way to properly describe the R–ratio is to employ the dispersion relation. Its re–expansion at $s \to \infty$ yields

$$R_{\text{appr}}^{(\ell)}(s) = 1 + \sum_{j=1}^{\ell} d_j \left[a_{\text{s}}^{(\ell)}(|s|)\right]^j - \sum_{j=1}^{\ell} d_j \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\pi^{2n}}{(2n+1)!} \times$$
$$\times \sum_{k_1=0}^{\ell-1} \ldots \sum_{k_{2n}=0}^{\ell-1} \left(\prod_{p=1}^{2n} B_{k_p}\right) \left[\prod_{t=0}^{2n-1} \Bigl(j + t + k_1 + \ldots + k_t\Bigr)\right] \times$$
$$\times \left[a_{\text{s}}^{(\ell)}(|s|)\right]^{j+2n+k_1+\ldots+k_{2n}}, \qquad \frac{\sqrt{s}}{\Lambda} > \exp\left(\frac{\pi}{2}\right) \simeq 4.81$$

■ **Nesterenko, Popov (2017); Nesterenko (2016, 2017)**

This re–expanded expression for the R–ratio:

- **constitutes the sum of $D_{\text{pert}}^{(\ell)}(|s|)$ and π^2–terms**
- **can be reduced to the form of power series in $a_{\text{s}}^{(\ell)}(|s|)$**
- **accurately approximates $R(s)$, if one retains many π^2–terms**

However, one commonly truncates $R^{(\ell)}_{\rm appr}(s)$ at a given order:

$$R^{(\ell)}_{\rm pert}(s) = 1 + \sum_{j=1}^{\ell} r_j \left[a_{\rm s}^{(\ell)}(|s|)\right]^j, \qquad r_j = d_j - \delta_j, \qquad B_j = \frac{\beta_j}{\beta_0^{j+1}},$$

$$\delta_1 = \delta_2 = 0, \quad \delta_3 = \frac{\pi^2}{3}d_1, \quad \delta_4 = \frac{\pi^2}{3}\left(\frac{5}{2}d_1 B_1 + 3d_2\right), \quad \delta_5 = \frac{\pi^2}{3}\left[\frac{3}{2}d_1\left(B_1^2 + 2B_2\right) + 7d_2B_1 + 6d_3\right] - \frac{\pi^4}{5}d_1,$$

$$\delta_6 = \frac{\pi^2}{3}\left[\frac{7}{2}d_1\left(B_1B_2 + B_3\right) + 4d_2\left(B_1^2 + 2B_2\right) + \frac{27}{2}d_3B_1 + 10d_4\right] - \frac{\pi^4}{5}\left(\frac{77}{12}d_1B_1 + 5d_2\right),$$

$$\delta_7 = \frac{\pi^2}{3}\left[4d_1\left(B_1B_3 + \frac{1}{2}B_2^2 + B_4\right) + 9d_2\left(B_1B_2 + B_3\right) + \frac{15}{2}d_3\left(B_1^2 + 2B_2\right) + 22d_4B_1 + 15d_5\right] - \frac{\pi^4}{5}\left[\frac{5}{6}d_1\left(17B_1^2 + 12B_2\right) + \frac{57}{2}d_2B_1 + 15d_3\right] + \frac{\pi^6}{7}d_1$$

- Bjorken (1989); Kataev, Starshenko (1995); Prosperi, Raciti, Simolo (2007); Nesterenko, Popov (2017); Nesterenko (2016, 2017)

This truncated re–expanded expression for the R–ratio:

- **contains infrared unphysical singularities**
- **is only valid for $\sqrt{s}/\Lambda = \sqrt{w} > \exp(\pi/2) \simeq 4.81$**
- **coefficients δ_j rapidly increase as the order j increases**
- **converges rather slowly when $\sqrt{s}/\Lambda$ approaches $\exp(\pi/2)$**

Table 1. Adler function perturbative expansion coefficients

n_f	d_1	d_2	d_3	d_4	d_5 (est.)
0	0.3636	0.2626	0.8772	2.3743	*5.40*
1	0.3871	0.2803	0.7946	2.1884	*4.70*
2	0.4138	0.3005	0.7137	2.1466	*3.74*
3	0.4444	0.3239	0.5593	1.9149	*2.52*
4	0.4800	0.3513	0.2868	1.3440	*1.16*
5	0.5217	0.3836	−0.1021	0.6489	*0.0256*
6	0.5714	0.4225	−0.7831	−0.8952	*0.267*

[SL] $$D^{(\ell)}_{\text{pert}}(Q^2) = 1 + \sum_{j=1}^{\ell} d_j \left[a_s^{(\ell)}(Q^2)\right]^j, \qquad a_s^{(\ell)}(Q^2) = \alpha_s^{(\ell)}(Q^2)\,\frac{\beta_0}{4\pi}$$

↓ [dispersion relations + re–expansion + truncation]

[TL] $$R^{(\ell)}_{\text{pert}}(s) = 1 + \sum_{j=1}^{\ell} r_j \left[a_s^{(\ell)}(|s|)\right]^j, \qquad r_j = d_j - \delta_j, \qquad \frac{\sqrt{s}}{\Lambda} > \exp\left(\frac{\pi}{2}\right)$$

Table 2. Coefficients embodying the π^2–terms contributions

n_f	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	δ_7 (est.)
0	0.0000	0.0000	1.1963	5.1127	20.455	69.081	*45.7*
1	0.0000	0.0000	1.2735	5.4298	18.880	56.819	*7.02*
2	0.0000	0.0000	1.3613	5.7583	17.118	48.532	*−35.7*
3	0.0000	0.0000	1.4622	6.0851	13.519	30.365	*−82.5*
4	0.0000	0.0000	1.5791	6.3850	6.910	−3.843	*−115.7*
5	0.0000	0.0000	1.7165	6.6090	−3.187	−45.692	*−83.0*
6	0.0000	0.0000	1.8799	6.6638	−21.168	−120.010	*142.5*

[SL] $$D^{(\ell)}_{\mathrm{pert}}(Q^2) = 1 + \sum_{j=1}^{\ell} d_j \left[a^{(\ell)}_{\mathrm{s}}(Q^2)\right]^j, \qquad a^{(\ell)}_{\mathrm{s}}(Q^2) = \alpha^{(\ell)}_{\mathrm{s}}(Q^2)\,\frac{\beta_0}{4\pi}$$

↓ [dispersion relations + re–expansion + truncation]

[TL] $$R^{(\ell)}_{\mathrm{pert}}(s) = 1 + \sum_{j=1}^{\ell} r_j \left[a^{(\ell)}_{\mathrm{s}}(|s|)\right]^j, \qquad r_j = d_j - \delta_j, \qquad \frac{\sqrt{s}}{\Lambda} > \exp\left(\frac{\pi}{2}\right)$$

Table 3. R–ratio perturbative expansion coefficients

n_f	$r_1 = d_1$	$r_2 = d_2$	$r_3 = d_3 - \delta_3$	$r_4 = d_4 - \delta_4$	$r_5 = d_5 - \delta_5$
0	0.3636	0.2626	−0.3191	−2.7383	*−15.1*
1	0.3871	0.2803	−0.4788	−3.2413	*−14.2*
2	0.4138	0.3005	−0.6476	−3.6116	*−13.4*
3	0.4444	0.3239	−0.9028	−4.1703	*−11.0*
4	0.4800	0.3513	−1.2923	−5.0409	*−5.75*
5	0.5217	0.3836	−1.8186	−5.9601	*3.21*
6	0.5714	0.4225	−2.6630	−7.5590	*21.4*

[SL] $$D^{(\ell)}_{\text{pert}}(Q^2) = 1 + \sum_{j=1}^{\ell} d_j \left[a_s^{(\ell)}(Q^2)\right]^j, \qquad a_s^{(\ell)}(Q^2) = \alpha_s^{(\ell)}(Q^2)\,\frac{\beta_0}{4\pi}$$

↓ [dispersion relations + re–expansion + truncation]

[TL] $$R^{(\ell)}_{\text{pert}}(s) = 1 + \sum_{j=1}^{\ell} r_j \left[a_s^{(\ell)}(|s|)\right]^j, \qquad r_j = d_j - \delta_j, \qquad \frac{\sqrt{s}}{\Lambda} > \exp\left(\frac{\pi}{2}\right)$$

Table 4. Relative weight of π^2–terms in the coefficients r_j

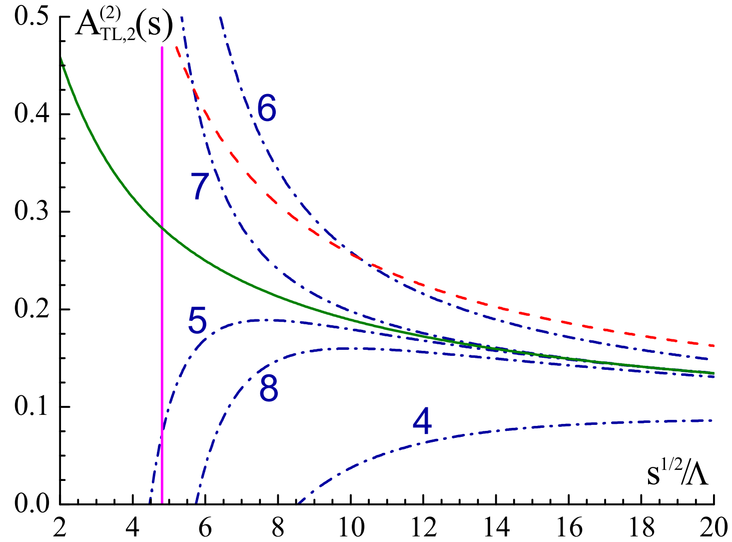
n_f	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$ (est.)
0	0.00 %	0.00 %	57.7 %	68.3 %	*79.1 %*
1	0.00 %	0.00 %	61.6 %	71.3 %	*80.1 %*
2	0.00 %	0.00 %	65.6 %	72.8 %	*82.1 %*
3	0.00 %	0.00 %	72.3 %	76.1 %	*84.3 %*
4	0.00 %	0.00 %	84.6 %	82.6 %	*85.6 %*
5	0.00 %	0.00 %	94.4 %	91.1 %	*99.2 %*
6	0.00 %	0.00 %	70.6 %	88.2 %	*98.8 %*

[SL] $$D^{(\ell)}_{\text{pert}}(Q^2) = 1 + \sum_{j=1}^{\ell} d_j \left[a_s^{(\ell)}(Q^2)\right]^j, \qquad a_s^{(\ell)}(Q^2) = \alpha_s^{(\ell)}(Q^2)\,\frac{\beta_0}{4\pi}$$

↓ [dispersion relations + re–expansion + truncation]

[TL] $$R^{(\ell)}_{\text{pert}}(s) = 1 + \sum_{j=1}^{\ell} r_j \left[a_s^{(\ell)}(|s|)\right]^j, \qquad r_j = d_j - \delta_j, \qquad \frac{\sqrt{s}}{\Lambda} > \exp\left(\frac{\pi}{2}\right)$$

Continuation of the two–loop strong running coupling squared into the timelike domain:



The functions are scaled by the factor of 10

$$\textbf{[SL]} \quad \left[a_{\rm S}^{(2)}(Q^2)\right]^2 \longrightarrow A^{(2)}_{\rm TL,\,2}(s) \quad \textbf{[TL]}$$

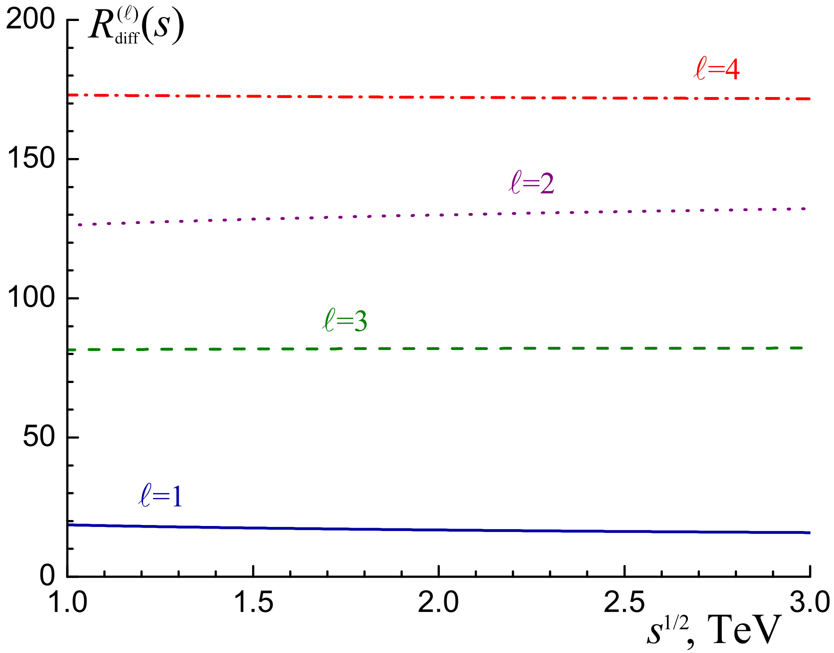
Its re–expansion for $\sqrt{s}/\Lambda > \exp(\pi/2)$**:**

$$A^{(2)}_{\rm TL,\,2}(s) \simeq \left[a_{\rm S}^{(2)}(|s|)\right]^2 - \frac{\pi^2}{\ln^4 w} + \frac{\pi^2}{\ln^5 w} B_1\bigl(4 \ln\ln w - 7/3\bigr) + \mathcal{O}\bigl(\ln^{-6} w\bigr)$$

However, at two–loop level all π^2**–terms are truncated, that gives a rather large error even at high energies. For example,** $\left[a_{\rm S}^{(2)}(|s|)\right]^2 \simeq 1.21 A^{(2)}_{\rm TL,\,2}(s)$ **at** $\sqrt{s}/\Lambda = 20$**, and to securely achieve** $10\,\%$ **accuracy one needs to include the** π^2**–terms up to** $\ln^{-7} w$**. Similarly, at** $\sqrt{s} = M_\tau$ **the relative difference between** $r^{(\ell)}(s)$ **and** $r^{(\ell)}_{\rm pert}(s)$ **is** $26\,\%$, $28\,\%$, $14\,\%$, $2\,\%$, **and** $7\,\%$ **for** $\ell = 1 \ldots 5$**.**

■ **Nesterenko (2016, 2017)**

Moreover, the ignorance of the higher–order π^2–terms may produce a considerable effect even at high energies:



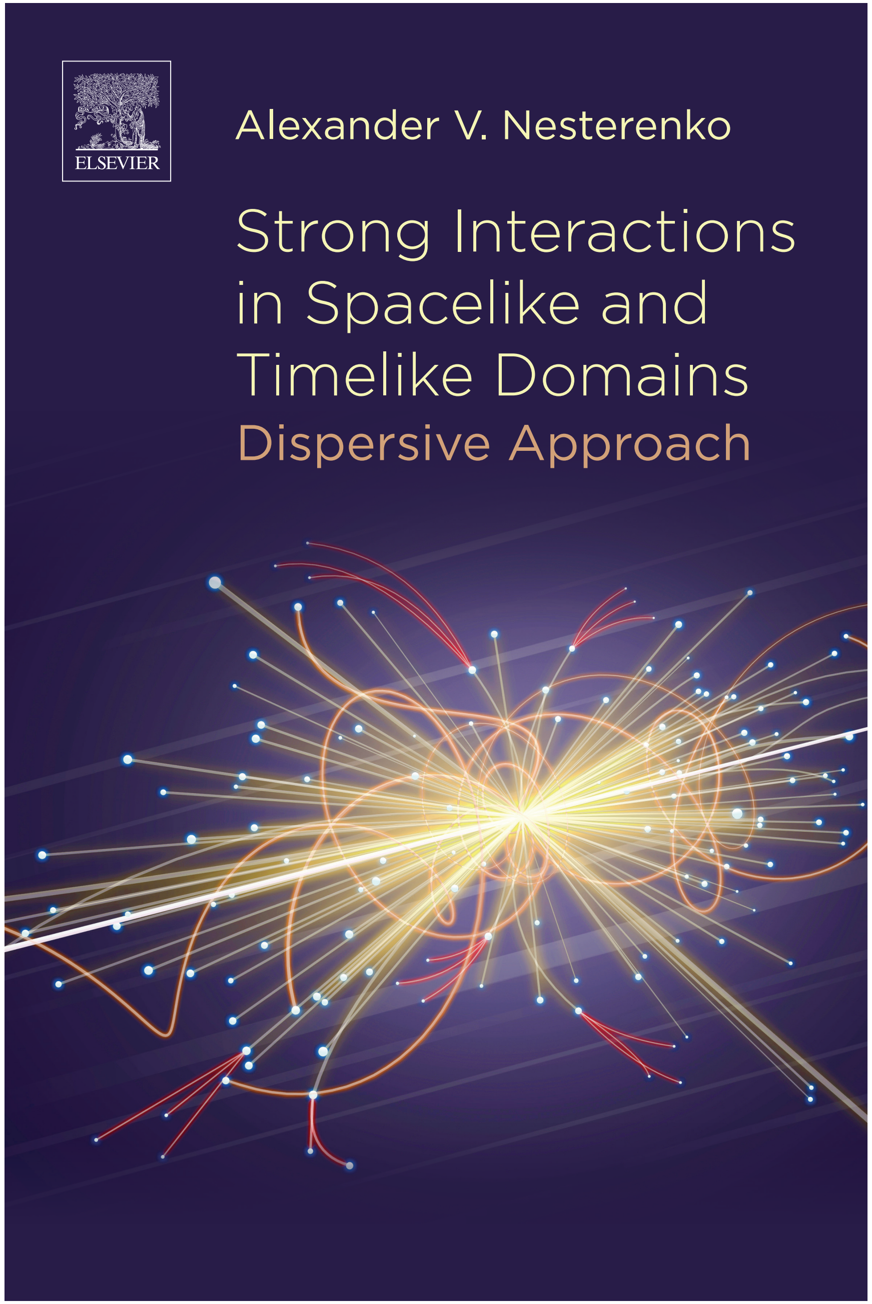
$$R^{(\ell)}_{\rm diff}(s) = \left| \frac{R^{(\ell)}_{\rm pert}(s) - R^{(\ell)}(s)}{R^{(\ell)}_{\rm pert}(s) - R^{(\ell+1)}_{\rm pert}(s)} \right| \times 100\,\%$$

Specifically, in the energy range planned for the CLIC experiment the effect of inclusion of the π^2–terms discarded in $R^{(\ell)}_{\rm pert}(s)$ is either comparable to or prevailing over the effect of inclusion of the next–order perturbative correction.

■ **Nesterenko (2017)**

SUMMARY

- **The integral representations for $\Pi(q^2)$, $R(s)$, and $D(Q^2)$ are derived in the framework of dispersive approach to QCD**
- **These representations merge the corresponding perturbative input with physical nonperturbative constraints**
- **Dispersive approach properly embodies SL $\longrightarrow$ TL effects**
- **Explicit expression for the perturbative spectral function valid at an arbitrary loop level is obtained**
- **The obtained results are in a good agreement with relevant lattice data and low–energy experimental predictions**
- **The developed approach yields reasonable assessments of the hadronic contributions to electroweak observables**



The detailed discussion of the presented results and other related topics can also be found in:

A.V. Nesterenko

Strong interactions in spacelike and timelike domains: Dispersive approach

Elsevier, Amsterdam, 2016

www.elsevier.com/books/isbn/9780128034392