Hadronic Corrections to $(g-2)_\mu$

via Schwinger's Sum Rule

Franziska Hagelstein (AEC Bern)

in coll. with Vladimir Pascalutsa (JGU Mainz)

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Dissecting the Hadronic Contributions to $(g-2)_\mu$ by Schwinger's Sum Rule

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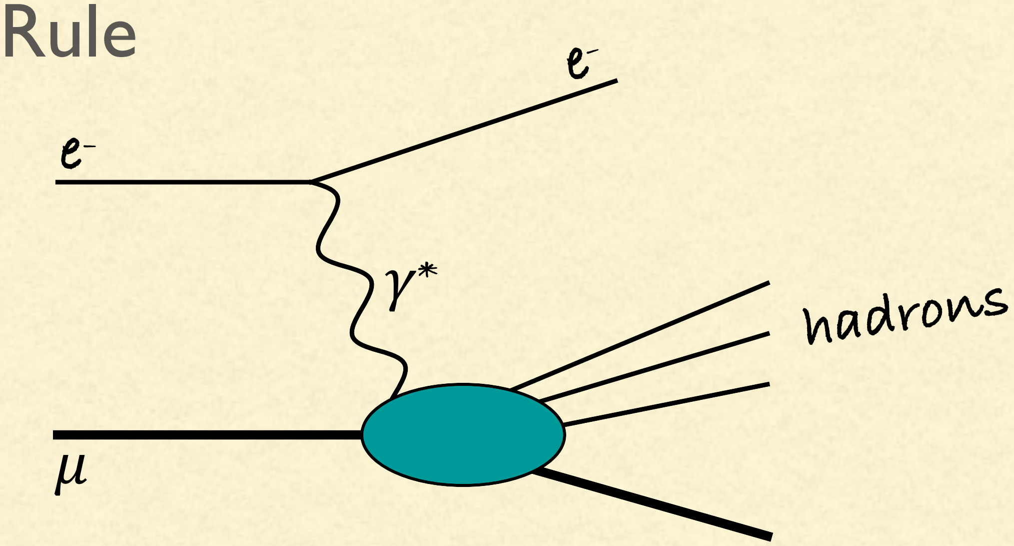
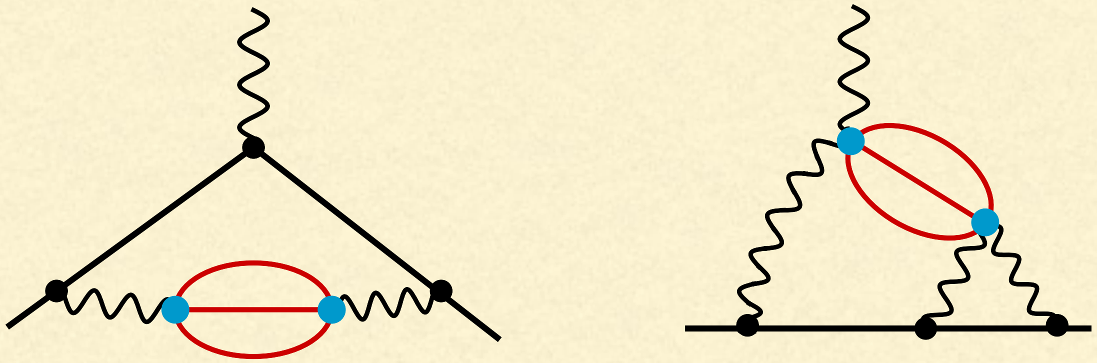
OUTLINE

- THE SCHWINGER SUM RULE $\varkappa = \frac{m^2}{\pi^2\alpha}\int_{\nu_0}^{\infty} \mathrm{d}\nu \left[\frac{\sigma_{LT}(\nu,Q^2)}{Q}\right]_{Q^2=0}$
- THE SCHWINGER TERM $\varkappa^{(1)} = \alpha/2\pi$
- HADRONIC VACUUM POLARIZATION
  - Reproducing the Standard Formula
  - Advantages of using the Schwinger Sum Rule
  - What is needed from Experiment: Inelastic Muon Spin Structure Functions
- OUTLOOK



MOTIVATION

- Uncertainty of the SM prediction for the muon anomaly $(g-2)_\mu$ is dominated by hadronic contributions (HVP and HLbL)



- HVP is calculated with a data-driven dispersive approach:

$$\varkappa^{\mathrm{HVP}} = \frac{\alpha}{\pi^2} \int_{4m_\pi^2}^{\infty} \frac{\mathrm{d}s}{s} \, \mathrm{Im}\, \Pi^{\mathrm{had}}(s) \, K(s/m^2)$$

$$\mathrm{Im}\, \Pi^{\mathrm{had}}(s) = \frac{s}{4\pi\alpha} \sigma(\gamma^* \to \text{anything})$$

F. Jegerlehner, Springer Tracts Mod. Phys. 274 (2017).

M. Davier, Nucl. Part. Phys. Proc. 287-288, 70 (2017) [arXiv:1612.02743 [hep-ph]].

MOTIVATION

$$\varkappa^{\text{HVP}} = \frac{\alpha}{\pi^2} \int_{4m_\pi^2}^{\infty} \frac{\mathrm{d}s}{s} \, \mathrm{Im}\, \Pi^{\text{had}}(s) \, K(s/m^2)$$

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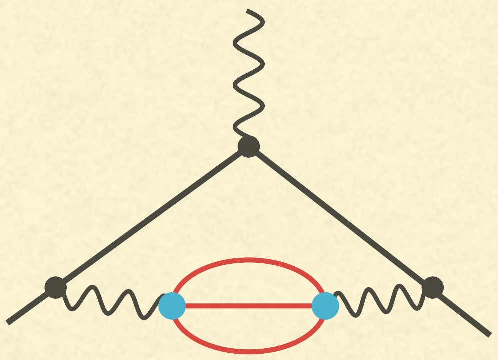
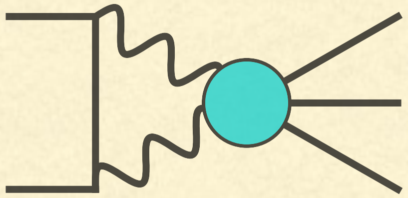


- Limitations of the standard approach to HVP:
  - Two-photon exchange corrections
  - QED radiative corrections

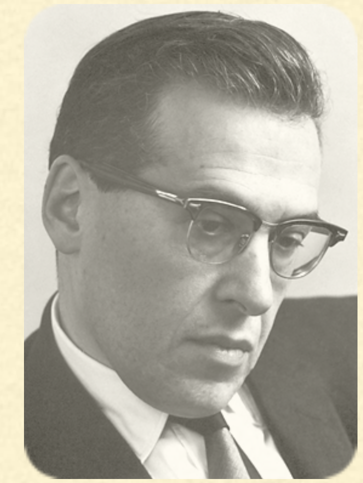


- For HLbL: no analogue of the simple dispersive formula
- In this talk: Schwinger sum rule as an alternative data-driven dispersive approach for both HVP and HLbL

THE SCHWINGER SUM RULE (1975)

J. S. Schwinger, Proc. Nat. Acad. Sci. 72, 1 (1975); ibid. 72, 1559 (1975) [Acta Phys. Austriaca Suppl. 14, 471 (1975)].

A. M. Harun ar-Rashid, Nuovo Cim. A 33, 447 (1976).



anomalous magnetic moment (a.m.m.) $\varkappa = 1/2\,(g-2)_\mu$

muon mass m

photon lab-frame energy ν and virtuality $Q^2 = -q^2$

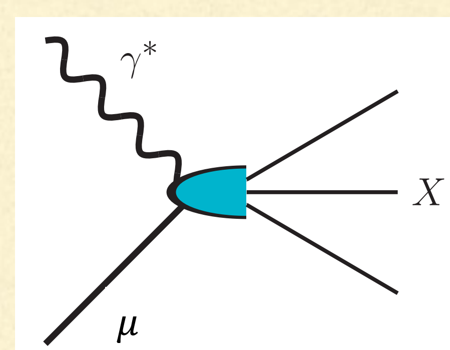
$$\varkappa = \frac{m^2}{\pi^2\alpha}\int_{\nu_0}^{\infty} \mathrm{d}\nu \left[\frac{\sigma_{LT}(\nu,Q^2)}{Q}\right]_{Q^2=0}$$

fine-structure constant $\alpha \approx 1/137$

photo-absorption threshold ν_0

longitudinal-transverse photo-absorption cross section σ_{LT}

- Cross sections for photo-absorption on muon:



inelastic cross section

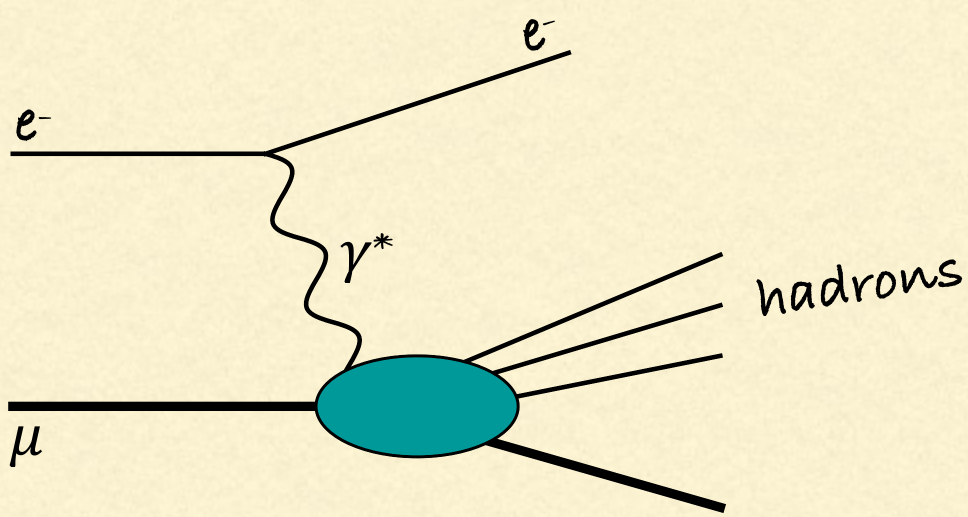
$X = \gamma\mu,\ \gamma\gamma\mu,\ \pi^0\mu,\ \gamma\pi^0\mu \ldots$

THE SCHWINGER SUM RULE (1975)

a.m.m.
$\varkappa = 1/2\,(g-2)_\mu$

$$\varkappa = \frac{m^2}{\pi^2\alpha}\int_{\nu_0}^{\infty} \mathrm{d}\nu \left[\frac{\sigma_{LT}(\nu,Q^2)}{Q}\right]_{Q^2=0}$$

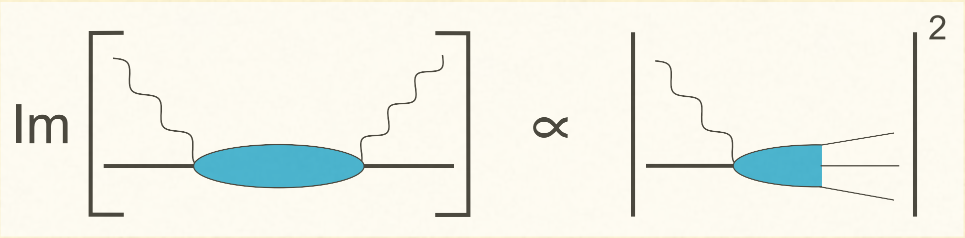
$$= \lim_{Q^2\to 0}\frac{8m^2}{Q^2}\int_0^{x_0}\mathrm{d}x\,[\bar{g}_1+\bar{g}_2](x,Q^2)$$



muon spin structure functions
g_1 and g_2

- Spin-dependent forward doubly-virtual Compton scattering:

$$T_A^{\mu\nu}(q,p) = -\frac{1}{M}\gamma^{\mu\nu\alpha}q_\alpha\,S_1(\nu,Q^2) + \frac{Q^2}{M^2}\gamma^{\mu\nu}S_2(\nu,Q^2)$$



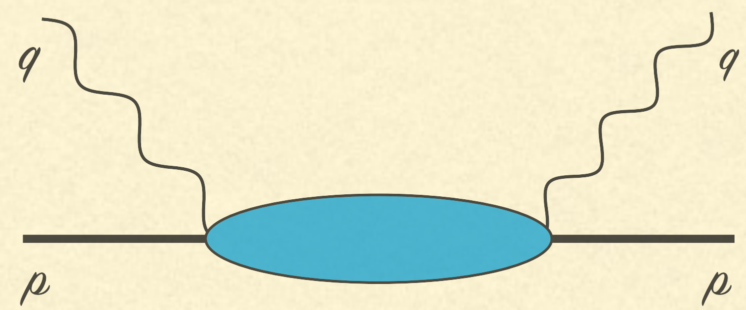
- Optical theorem:

$$\mathrm{Im}\,S_1(\nu,Q^2) = \frac{4\pi^2\alpha}{\nu}\,g_1(x,Q^2) = \frac{M\nu^2}{\nu^2+Q^2}\left[\frac{Q}{\nu}\sigma_{LT}+\sigma_{TT}\right](\nu,Q^2)$$

$$\mathrm{Im}\,S_2(\nu,Q^2) = \frac{4\pi^2\alpha M}{\nu^2}\,g_2(x,Q^2) = \frac{M^2\nu}{\nu^2+Q^2}\left[\frac{\nu}{Q}\sigma_{LT}-\sigma_{TT}\right](\nu,Q^2)$$

THE GDH AND BC SUM RULES

- Sum rules are model-independent relations based on very general principles:
  - Analyticity/causality and crossing symmetry (dispersion relations), and unitarity (optical theorem)



- Sum rules of Compton scattering off a spin-1/2 particle:

$$(1+\varkappa)\varkappa = \frac{m^2}{\pi^2\alpha}\int_{\nu_0}^{\infty} \mathrm{d}\nu \left[\frac{\sigma_{LT}}{Q} - \frac{\sigma_{TT}}{\nu}\right]_{Q^2=0}$$

Burkhardt—Cottingham sum rule (1970) $\int_0^1 \mathrm{d}x\, g_2(x,Q^2) = 0$

$\ominus$ $$\varkappa^2 = -\frac{m^2}{\pi^2\alpha}\int_{\nu_0}^{\infty} \mathrm{d}\nu\, \frac{\sigma_{TT}(\nu)}{\nu}$$

Gerasimov—Drell—Hearn sum rule (1966)

$$\varkappa = \frac{m^2}{\pi^2\alpha}\int_{\nu_0}^{\infty} \mathrm{d}\nu \left[\frac{\sigma_{LT}(\nu,Q^2)}{Q}\right]_{Q^2=0}$$

Schwinger sum rule (1975)

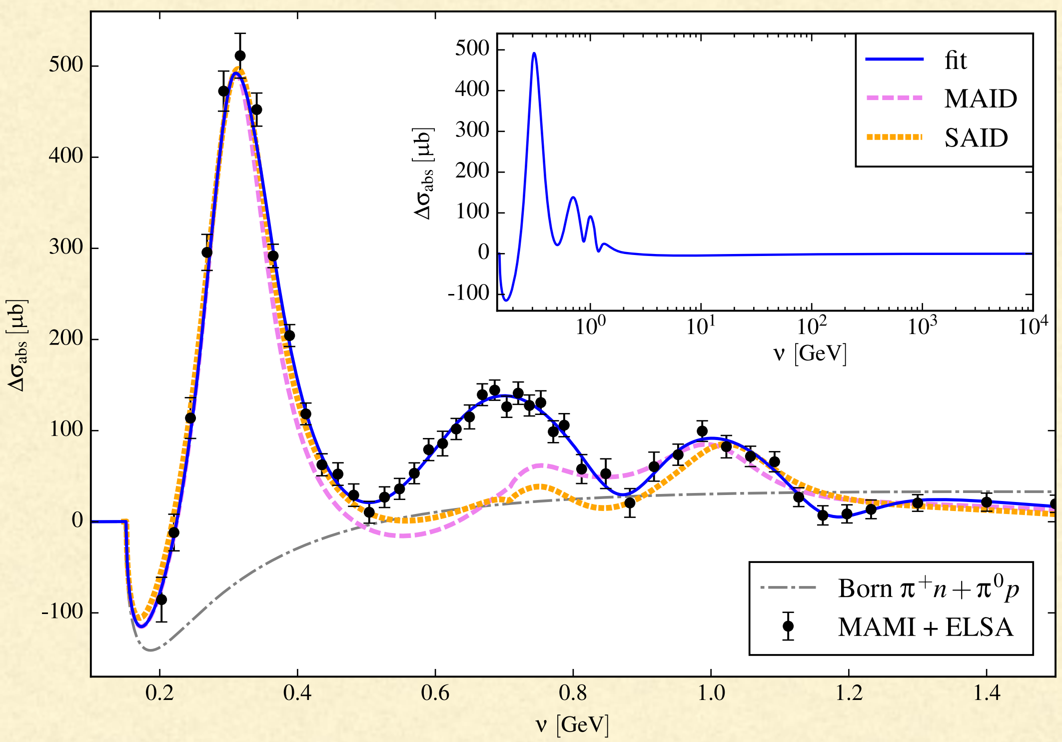
THE A.M.M. OF THE PROTON

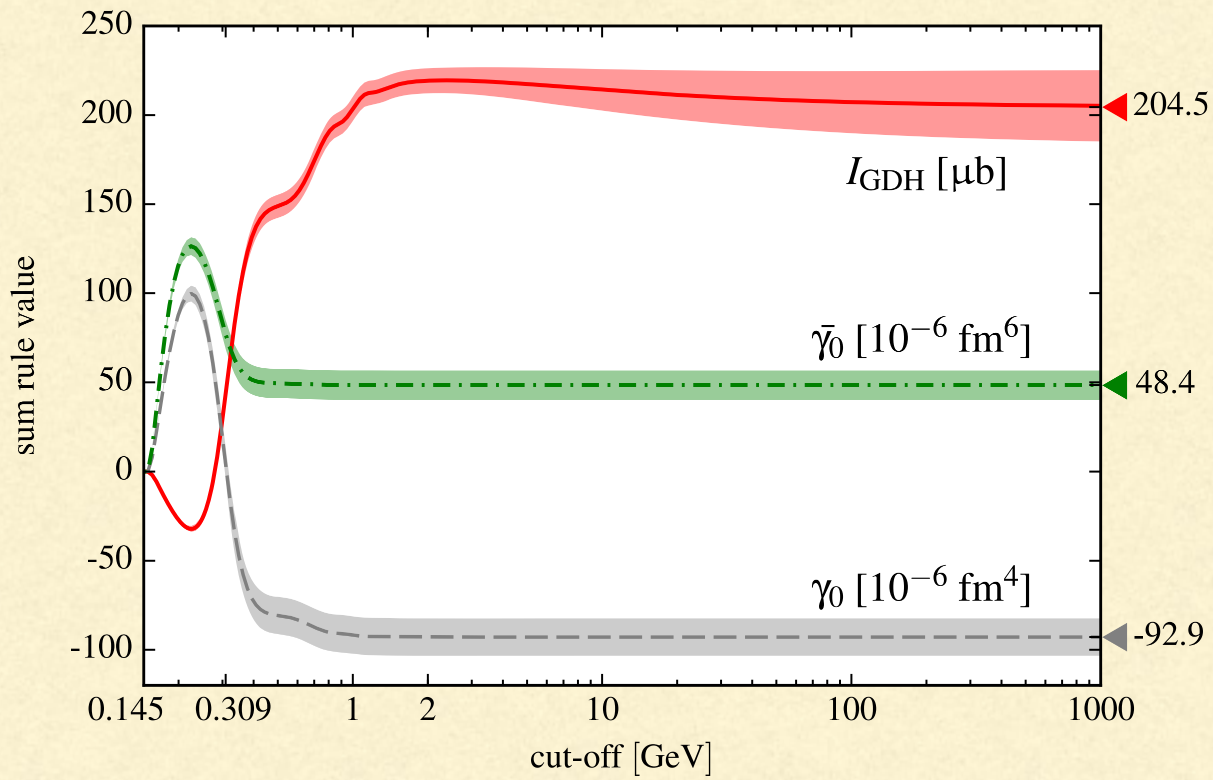
Gerasimov—Drell—Hearn sum rule

$$I_{\mathrm{GDH}} = \frac{2\pi^2\alpha}{m^2}\varkappa^2 = -2\int_{\nu_0}^{\infty} \mathrm{d}\nu\, \frac{\sigma_{\mathrm{TT}}(\nu)}{\nu}$$

$\varkappa_p \approx 1.7929$ and

$I_{\mathrm{GDH}} = 204.784481\ \mu b$ [CODATA]





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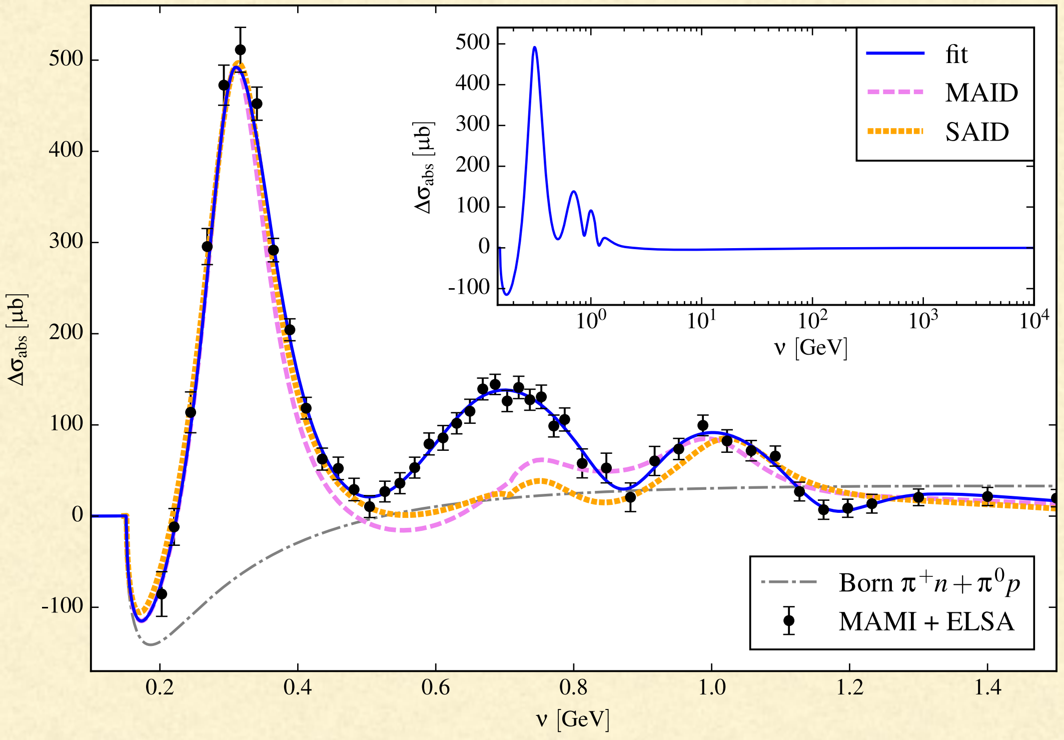
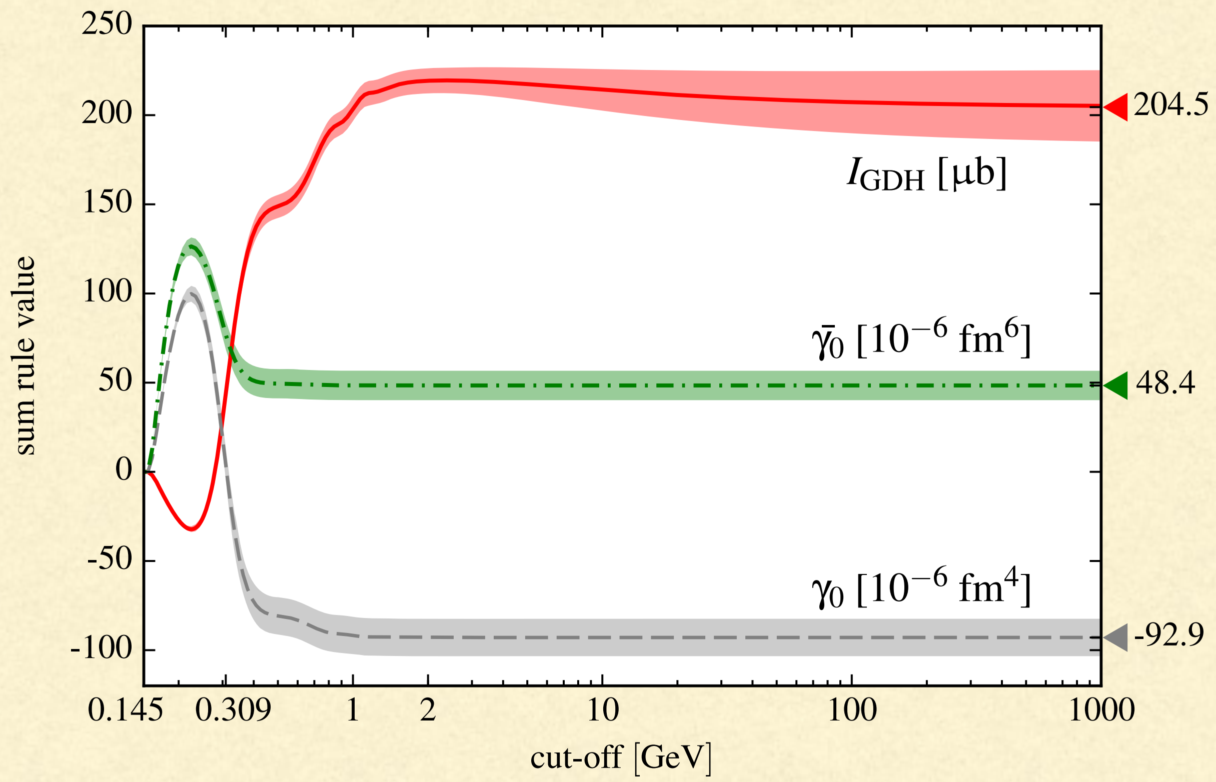
$\varkappa_\mu \approx 0.0011659209(6)$ [BNL]

- GDH sum rule for the muon:
  - huge cancelation requires measurements with incredible accuracy
    - r.h.s.: HVP starts at $\mathcal{O}(\alpha^2)$, I_{GDH} starts at $\mathcal{O}(\alpha^5)$
    - l.h.s.: hadronic photo-production cross section starts at $\mathcal{O}(\alpha^3)$

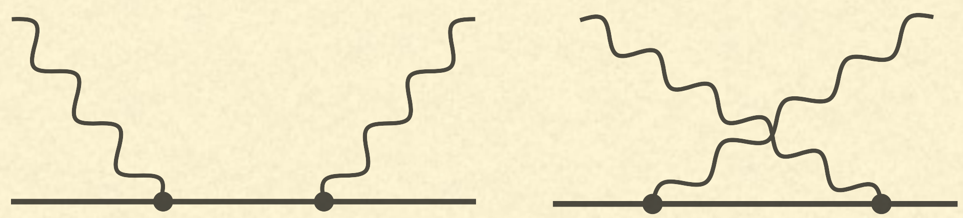




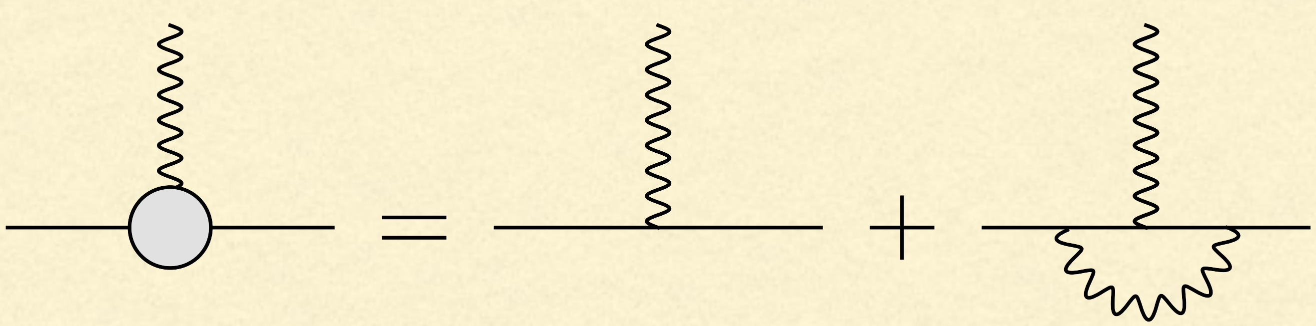
THE SCHWINGER TERM

- Schwinger sum rule: $\varkappa = \frac{m^2}{\pi^2 \alpha} \int_{\nu_0}^{\infty} \mathrm{d}\nu \left[\frac{\sigma_{LT}(\nu, Q^2)}{Q}\right]_{Q^2=0}$
- Input: longitudinal-transverse photo-absorption cross section

tree-level QED
Compton scattering



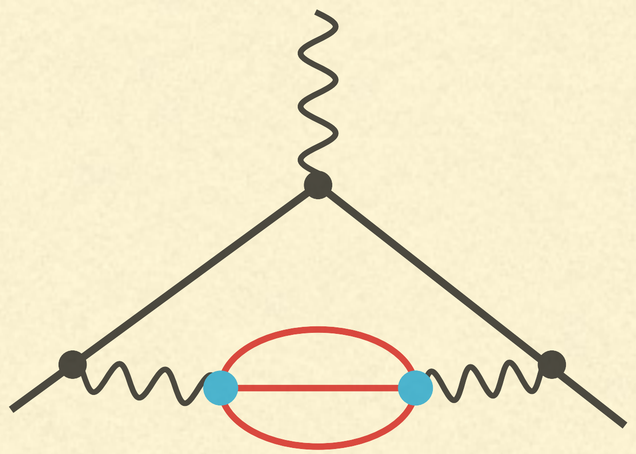
$$\sigma_{LT}^{\gamma^*\mu\to\gamma\mu}(\nu, Q^2) = \frac{\pi\alpha^2 Q\,(s-m^2)^2}{4m^3\nu^2\left(\nu^2+Q^2\right)}\left(-2 - \frac{m(m+\nu)}{s} + \frac{3m+2\nu}{\sqrt{\nu^2+Q^2}}\,\mathrm{arccoth}\frac{m+\nu}{\sqrt{\nu^2+Q^2}}\right)$$



$F_2(0) = \varkappa$ $\varkappa^{(0)} = 0$ $\varkappa^{(1)} = \alpha/2\pi$

HVP: STANDARD FORMULA

- Hadronic vacuum polarization: 2 Data-driven approaches based on dispersion theory
  - A) Standard Formula
  - B) Schwinger Sum Rule



A

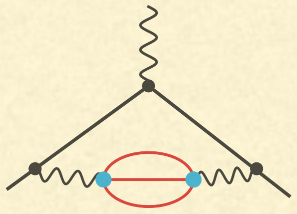
$$\varkappa^{\text{HVP}} = \frac{\alpha}{\pi^2}\int_{4m_\pi^2}^{\infty} \frac{\mathrm{d}s}{s}\, \operatorname{Im}\Pi^{\text{had}}(s) \int_0^1 \mathrm{d}x\, \frac{x^2(1-x)}{x^2+(1-x)(s/m^2)}$$

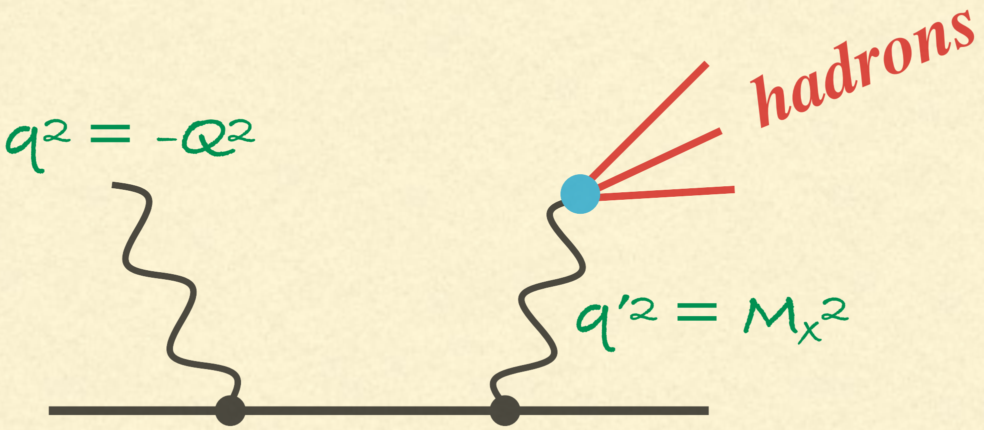
$$\operatorname{Im}\Pi^{\text{had}}(s) = \frac{s}{4\pi\alpha}\,\sigma(\gamma^* \to \text{anything})$$

↑ photon selfenergy

↑ decay rate of a virtual timelike photon into hadrons

HVP: SCHWINGER SUM RULE





+ crossed diagram

- Cross section of hadron production through timelike Compton scattering:

factories into: $$\sigma(\gamma\mu \to \mu X) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \frac{\mathrm{d}M_X^2}{M_X^2}\, \sigma(\gamma\mu \to \gamma^*\mu)\, \mathrm{Im}\,\Pi_X(M_X^2)$$

timelike Compton scattering — virtual-photon decay into hadrons

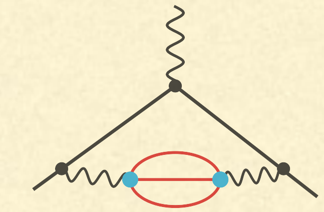
- Timelike Compton scattering cross section:

$$\left[\frac{\sigma_{LT}^{\gamma\mu\to\gamma^*\mu}(\nu,Q^2)}{Q}\right]_{Q^2=0} = \frac{\pi\alpha^2}{2m^2\nu^3}\left[-(5s+m^2+M_X^2)\lambda + (s+2m^2-2M_X^2)\log\frac{\beta+\lambda}{\beta-\lambda}\right]$$

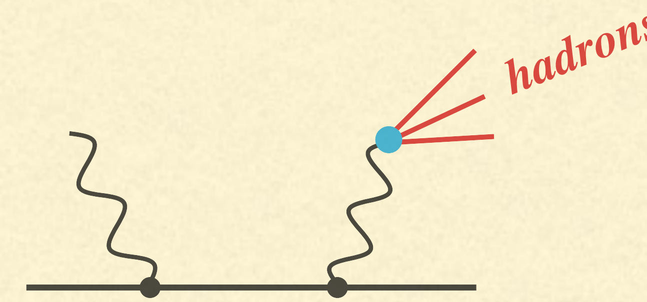
$$\beta = (s+m^2-M_X^2)/2s \qquad s = m^2+2m\nu$$

$$\lambda = (1/2s)\sqrt{[s-(m+M_X)^2]\,[s-(m-M_X)^2]}$$

HVP: SCHWINGER SUM RULE



- HVP from the Schwinger sum rule with the cross section of hadron production through timelike Compton scattering:



$$\varkappa = \frac{m^2}{\pi^2 \alpha} \int_{4m_\pi^2}^{\infty} \mathrm{d}M_X^2 \int_{\nu_0}^{\infty} \mathrm{d}\nu \left[\frac{1}{Q} \frac{\mathrm{d}\sigma_{LT}^{\gamma\mu\to\mu X}(\nu, Q^2)}{\mathrm{d}M_X^2} \right]_{Q^2=0}$$

$$= \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} \mathrm{d}M_X^2 \, \frac{\mathrm{Im}\,\Pi^{\mathrm{had}}(M_X^2)}{M_X^2} \, \frac{m^2}{\pi^2 \alpha} \int_{\nu_0}^{\infty} \mathrm{d}\nu \left[\frac{\sigma_{LT}^{\gamma\mu\to\gamma^*\mu}(\nu, Q^2)}{Q} \right]_{Q^2=0}$$

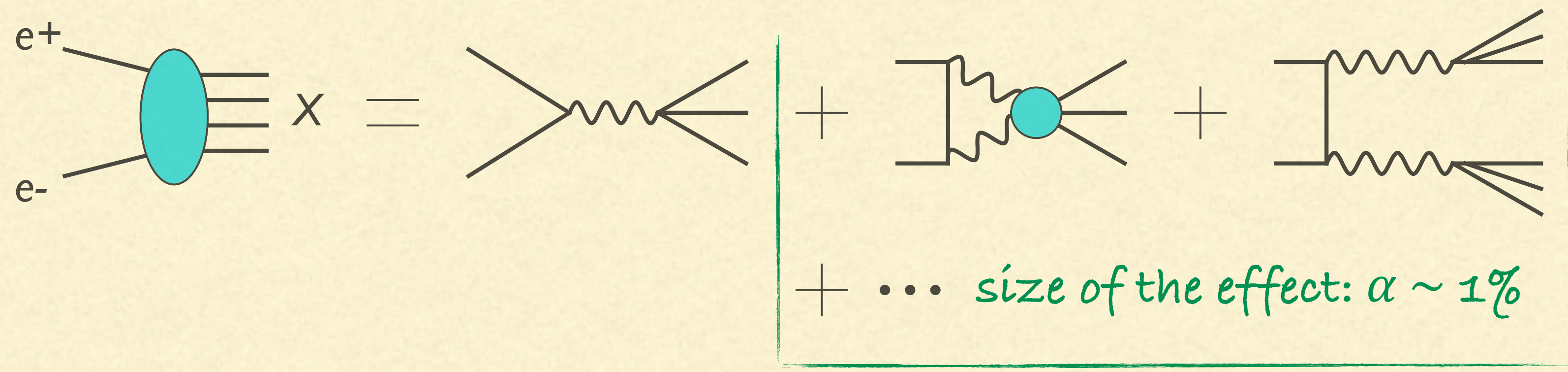
kernel function: $= \frac{\alpha}{\pi} K(M_X^2/m^2) \equiv \frac{\alpha}{\pi} \int_0^1 \mathrm{d}x \, \frac{x^2(1-x)}{x^2 + (1-x)(M_X^2/m^2)}$

for $M_x=0$, we find $K(0)=1/2$, and therefore the Schwinger term: $\varkappa^{(1)} = \alpha/2\pi$

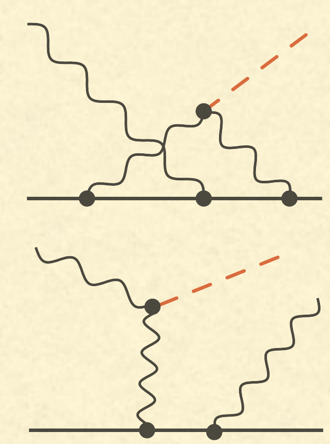
- Schwinger sum rule can reproduces the HVP standard formula

$$\varkappa^{\mathrm{HVP}} = \frac{\alpha}{\pi^2} \int_{4m_\pi^2}^{\infty} \frac{\mathrm{d}s}{s} \, \mathrm{Im}\,\Pi^{\mathrm{had}}(s) \int_0^1 \mathrm{d}x \, \frac{x^2(1-x)}{x^2 + (1-x)(s/m^2)}$$

LIMITATIONS & ADVANTAGES



- Limitations of the standard approach:
  - Two-photon exchange corrections
  - QED radiative corrections
- Advantages of the Schwinger sum rule:
  - Different mechanisms contribute and will be included in the measured cross section (… of course there is no data yet!)
  - No adjustment of the Schwinger sum rule needed for different mechanisms
  - No separation of radiative corrections necessary — as long as there are hadrons in the final state

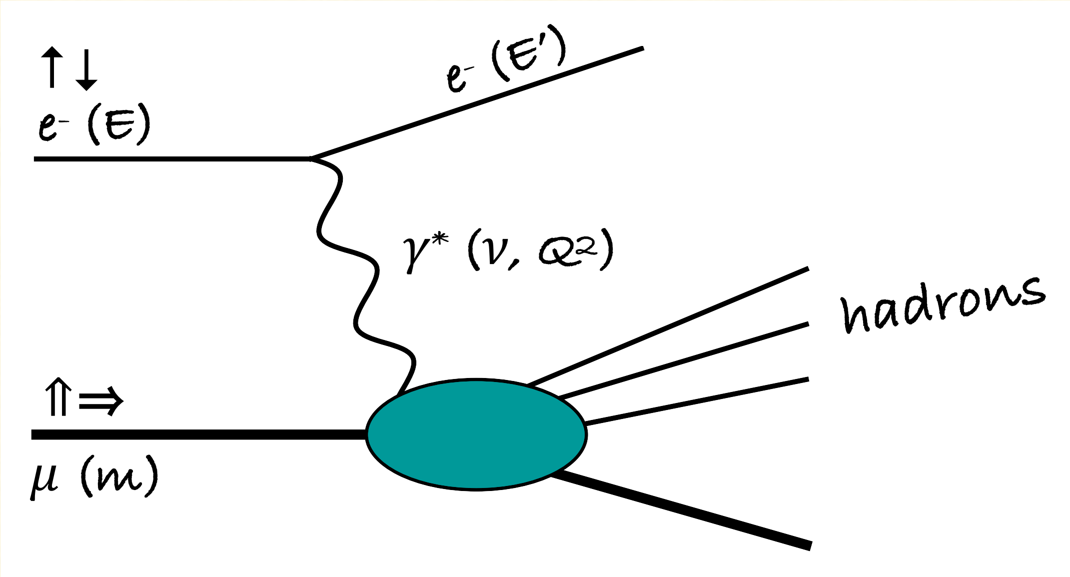


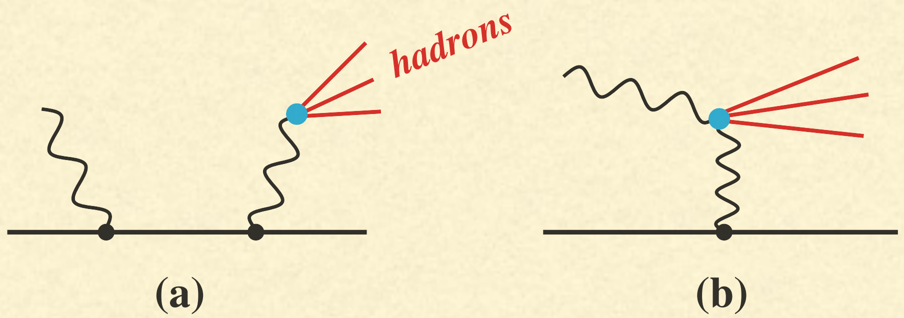
MUON STRUCTURE FUNCTIONS

- Muon spin structure functions measured in inelastic electron-muon scattering:
  - $\underset{\Uparrow}{\mu}e \to \underset{\uparrow}{\mu}e$ + hadrons

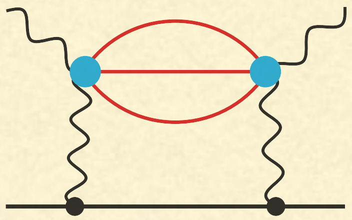
$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E'\mathrm{d}\Omega}(\downarrow\Uparrow - \uparrow\Uparrow) = \frac{4\alpha^2}{mQ^2}\frac{E'}{\nu E}\left[(E+E'\cos\theta)\,g_1(x,Q^2) - \frac{Q^2}{\nu}\,g_2(x,Q^2)\right]$$

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E'\mathrm{d}\Omega}(\downarrow\Rightarrow - \uparrow\Rightarrow) = \frac{4\alpha^2\sin\theta}{mQ^2}\frac{E'^2}{\nu^2 E}\left[\nu\,g_1(x,Q^2) - 2E\,g_2(x,Q^2)\right]$$





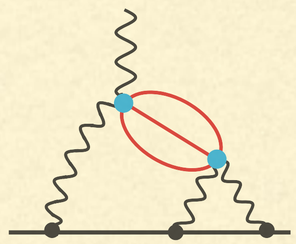
- Hadron photo-production off the muon:
  - (a) timelike Compton scattering
  - (b) Primakoff effect



- HLbL contribution to Compton scattering

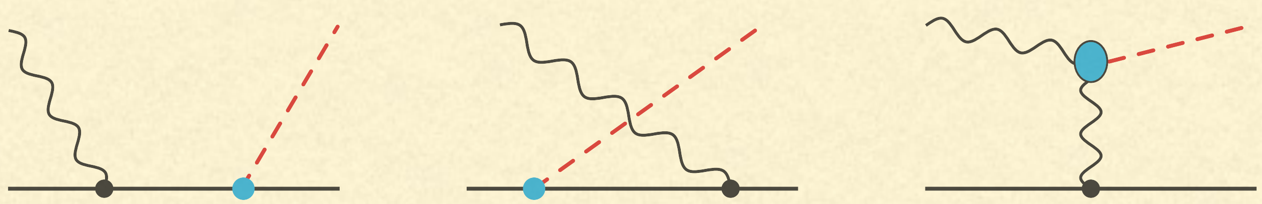
HADRONIC LIGHT-BY-LIGHT SCATTERING

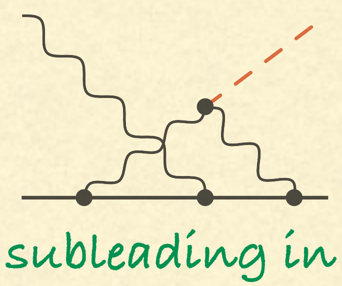
- HLbL is more complicated to calculate as it usually involves dispersion relations for 3- and/or 4-point functions



- Different Channels:

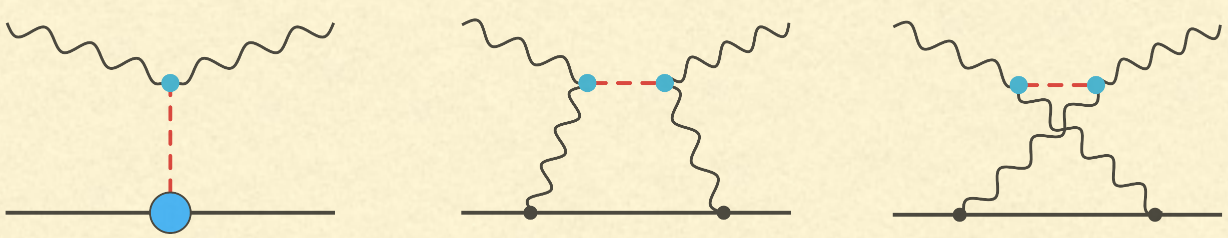
I. Hadron photo-production channels





- Clear relation to observables with no assumptions required

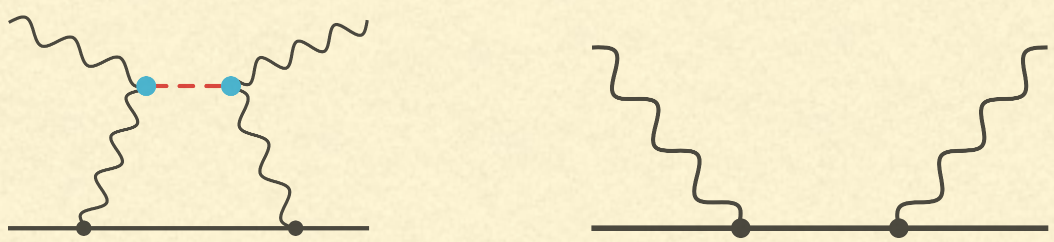
II. Electromagnetic channels



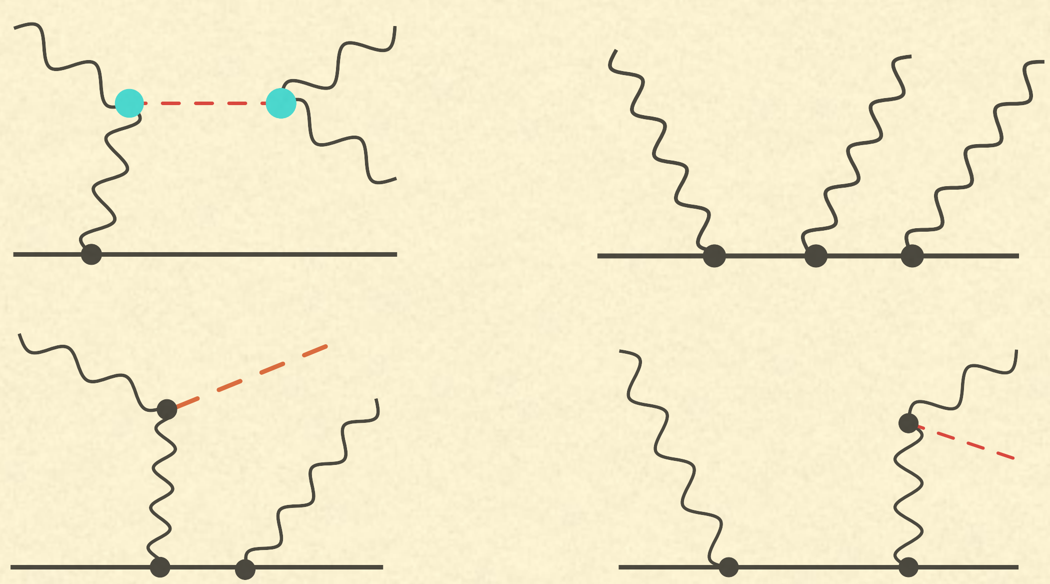
HADRONIC LIGHT-BY-LIGHT SCATTERING

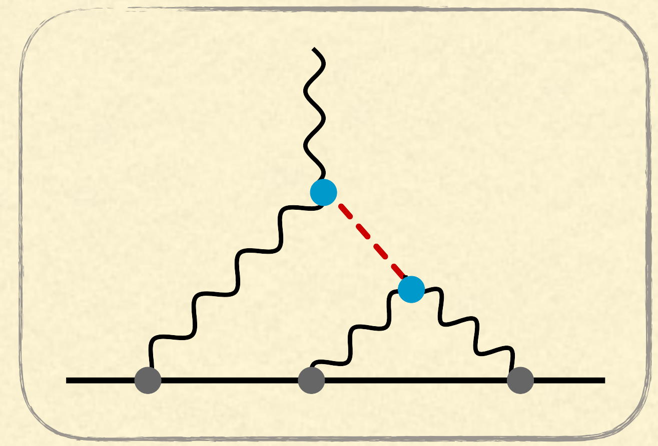
I. Hadron photo-production channel

II. Electromagnetic channels



III. Other possible contributions





pseudo-scalar pole contribution

- No doubly-virtual transition form factors needed, only $F_{\pi\gamma\gamma^*}$
- Reduced number of loops

OUTLOOK & CONCLUSIONS

- Schwinger sum rule is an exact formula for data-driven evaluation of hadronic contributions to the muon anomaly (both HVP and HLbL)
- Muon spin structure functions could in principle be measured in inelastic electron-muon scattering (with polarized muons)
  - polarized electron-muon collider
  - fixed-target μ-on-e scattering
- Hadron photo-production contribution can be measured directly
- HLbL contribution to the e.m. channels is easier to calculate than the standard HLbL, because of less virtual photons involved (only $F_{\pi\gamma\gamma^*}$ is needed) and a reduced number of loops

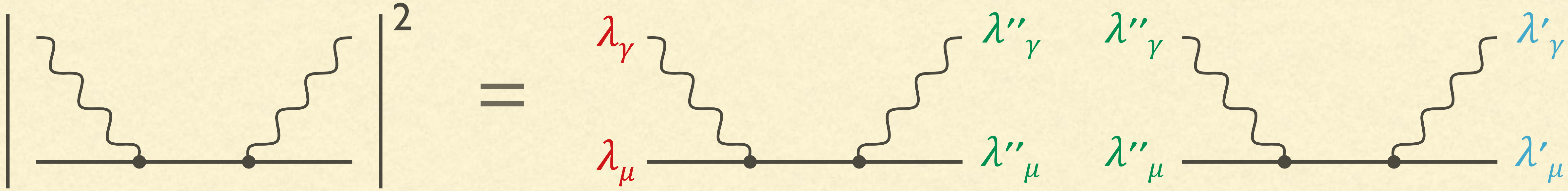
BACK-UP SLIDES

THE CROSS SECTION σ_{LT}

- Example: tree-level QED Compton scattering cross section

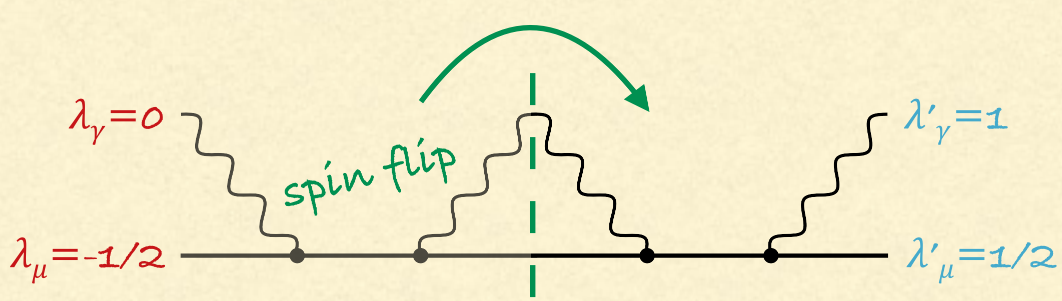
$$\mathrm{d}\sigma_{\lambda'_\gamma\lambda'_\mu\lambda_\gamma\lambda_\mu} = (2\pi)^4\delta^{(4)}(p_f - p_i)\sum_{\lambda''_\gamma,\lambda''_\mu}\frac{\mathcal{M}^\dagger_{\lambda'_\gamma\lambda'_\mu\lambda''_\gamma\lambda''_\mu}\mathcal{M}_{\lambda''_\gamma\lambda''_\mu\lambda_\gamma\lambda_\mu}}{4I}\prod_a\frac{\mathrm{d}^3p'_a}{(2\pi)^3 2E'_a},$$

with conserved helicity: $H=\lambda'_\gamma-\lambda'_\mu=\lambda_\gamma-\lambda_\mu$

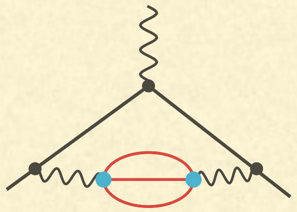


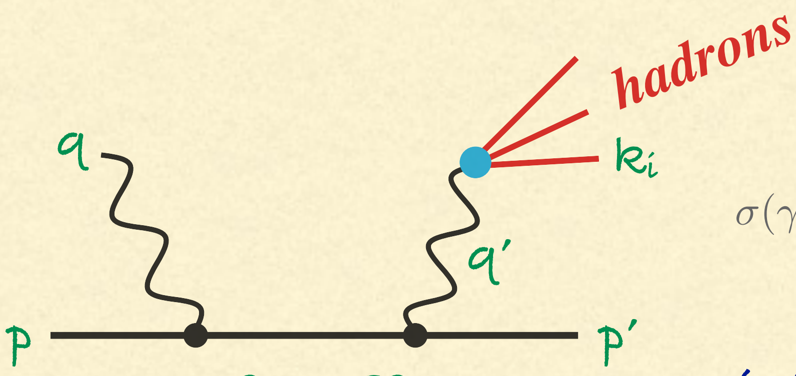
- helicity difference photo-absorption cross section: $\sigma_{TT} = {}^1\!/_2\,(\sigma_{1/2} - \sigma_{3/2})$
- longitudinal-transverse photo-absorption cross section:

$\gamma^*(\lambda_\gamma=0) + \mu(\lambda_\mu=-1/2) \rightarrow \gamma(\lambda'_\gamma=1) + \mu(\lambda'_\mu=1/2)$



TIME-LIKE CS & PHOTON DECAY





$q^2 = -Q^2$

$q'^2 = M_X^2$

+ crossed diagram

Λ^ν: virtual-photon decay vertex

$$\sigma(\gamma\mu \to \mu X) = \frac{(2\pi)^4}{4I} \int \mathrm{d}^4 q' \int \prod_i \frac{\mathrm{d}^3 \boldsymbol{k_i}}{2E_{k_i}(2\pi)^3} \int \frac{\mathrm{d}^3 \boldsymbol{p'}}{2E_{p'}(2\pi)^3} \left[\frac{\Lambda^{\dagger\mu}\Lambda^\nu \rho_{\mu\nu}}{(-q'^2)^2}\right] \delta^4(q' - \sum_i k_i)\, \delta^4(p + q - p' - q')$$

initial flux factor $I^2 = (p \cdot q)^2 - p^2 q^2$

phase space of the final state

$\rho_{\mu\nu}$: squared matrix element of timelike CS

- Virtual-photon decay width into hadronic state X:

$$[\Gamma(\gamma^* \to X)]^{\mu\nu} = \int \prod_i \frac{\mathrm{d}^3 \boldsymbol{k_i}}{2E_{k_i}(2\pi)^3} \frac{\Lambda^{\dagger\mu}\Lambda^\nu}{2E_{q'}} (2\pi)^4 \delta^4(q' - \sum_i k_i)$$
$$= -\frac{1}{\sqrt{q'^2}} (q'^2 g^{\mu\nu} - q'^\mu q'^\nu) \operatorname{Im} \Pi_X(q'^2)$$

$\operatorname{Im} \Pi_X$: contribution of state X to the VP

- Combine into: $\sigma(\gamma\mu \to \mu X) = -\frac{1}{2I} \int \mathrm{d}^4 q' \int \frac{\mathrm{d}^3 \boldsymbol{p'}}{2E_{p'}(2\pi)^3} \rho^\mu_\mu \frac{\operatorname{Im} \Pi_X(q'^2)}{q'^2} \delta^4(p + q - p' - q')$

- Final factorized cross section: $\sigma(\gamma\mu \to \mu X) = \frac{1}{\pi} \int_{4m_\pi^2}^\infty \frac{\mathrm{d}M_X^2}{M_X^2} \sigma(\gamma\mu \to \gamma^*\mu) \operatorname{Im} \Pi_X(M_X^2)$