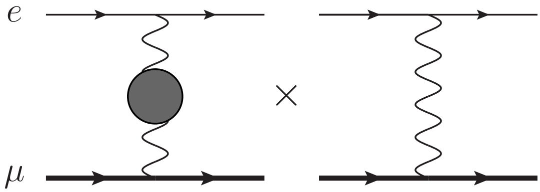
Hadronic NLO contributions to μ-e scattering

Matteo Fael - Universität Siegen

MITP Mainz, 22.2.2018

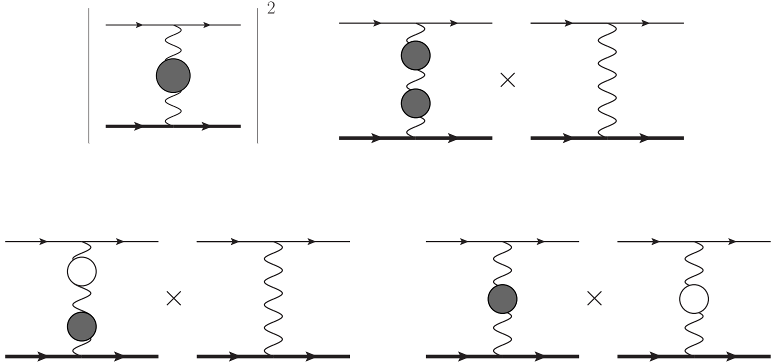
Hadronic LO Contribution



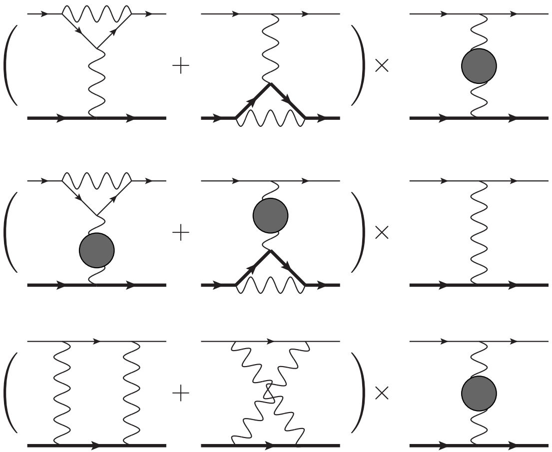
How to calculate these diagrams of $O(\alpha^4)$?

How to calculate these diagrams of $O(\alpha^4)$ only with space-like data.

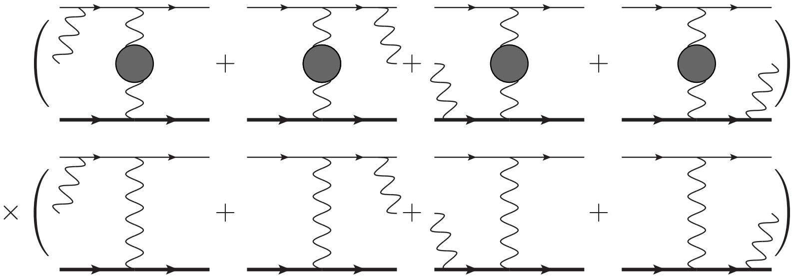
Class I



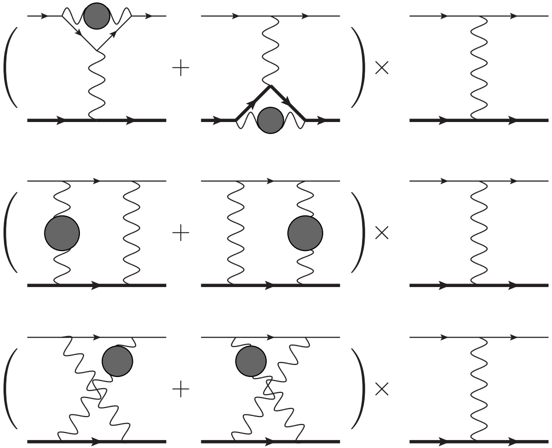
Class II



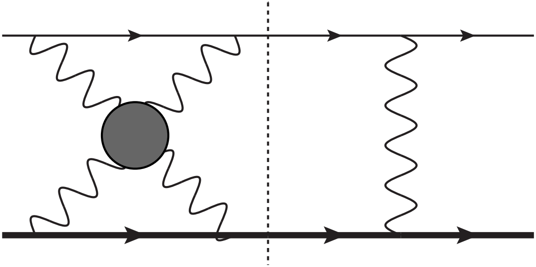
Class III

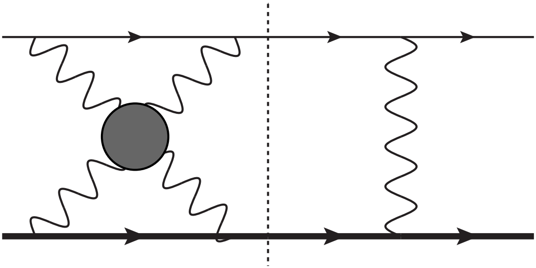


Class IV




BUT THE LBL CONTRIBUTION...
THERE IS NO LBL!
imgflip.com





is of $O(\alpha^5)$.

The Dispersive Approach

Dispersion Relation + Optical Theorem

$$\Pi^{\rm had}(q^2) = -\frac{q^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{dz}{z}\frac{{\rm Im}\Pi^{\rm had}(z)}{q^2 - z + i\varepsilon}$$

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$$\frac{-ig^{\mu\nu}}{q^2} \to \frac{-ig^{\mu\nu}}{q^2}\Pi^{\text{had}}(q^2)$$

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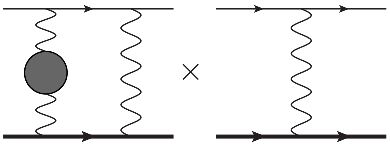
$$\frac{-ig^{\mu\nu}}{q^2} \to \frac{-ig^{\mu\nu}}{q^2}\Pi^{\mathrm{had}}(q^2) \to -\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{dz}{z}\,\mathrm{Im}\Pi^{\mathrm{had}}(z)\left[\frac{-ig^{\mu\nu}}{q^2 - z}\right]$$



Hadronic NNLO Bhabha scattering:
Actis, Czakon, Gluza, Riemann, PRL 100 (2008) 131602; Actis, Gluza, Riemann, Nucl. Phys. Proc. Suppl. 183 (2008) 174; Kühn, Uccirati, NPB 806 (2009) 300.

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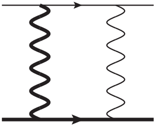
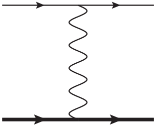
$$\frac{-ig^{\mu\nu}}{q^2} \to \frac{-ig^{\mu\nu}}{q^2}\Pi^{\rm had}(q^2) \to -\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{dz}{z}\,\mathrm{Im}\Pi^{\rm had}(z)\,\left[\frac{-ig^{\mu\nu}}{q^2 - z}\right]$$

$$-\frac{\alpha}{3\pi}\int_{4m_\pi^2}^{\infty}\frac{dz}{z}R^{\rm had}(z)$$  $\times$ 

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Numerical Implementation: QED NLO

With the FeynArts + FormCalc framework:

- FeynArts generates Feynman diagrams in QED.
- FormCalc calculates and simplifies tree-level and one-loop diagrams.
- FormCalc exports $|\mathcal{M}|^2$ as Fortran code.

T. Hahn, Comput. Phys. Commun. 140 (2001) 418;
T. Hahn, S. Passehr and C. Schappacher, J. Phys. Conf. Ser. 762 (2016) 012065

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The Montecarlo code:

- Full dependence on m_e and m_μ.
- `Collier` evaluates one-loop functions.

  Denner, Dittmaier, Hofer, Comput. Phys. Commun. **212** (2017) 220
- Soft singularities with FKS subtraction.
  Frixione, Kunszt, Signer, NPB 467 (1996) 399
  Frederix, Frixione, Maltoni, Stelzer, JHEP 0910 (2009) 003

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Very good agreement with Carloni Calame et al.

Numerical Implementation: Hadronic NLO

With the `FeynArts` + `FormCalc` framework:

- `QEDv2.mod`: QED model + one massive photon with mass z.
- For class I,II, III back-substitute the dispersion relation.
- $|\mathcal{M}|^2$ of class IV depends on z.

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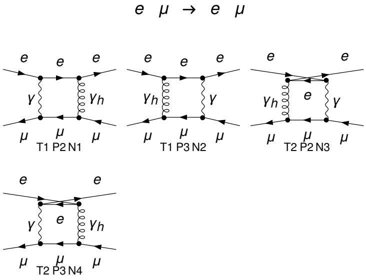

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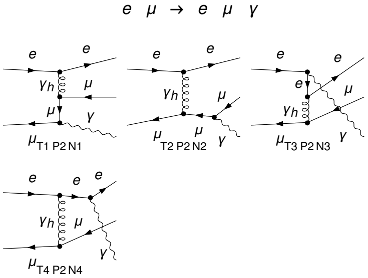

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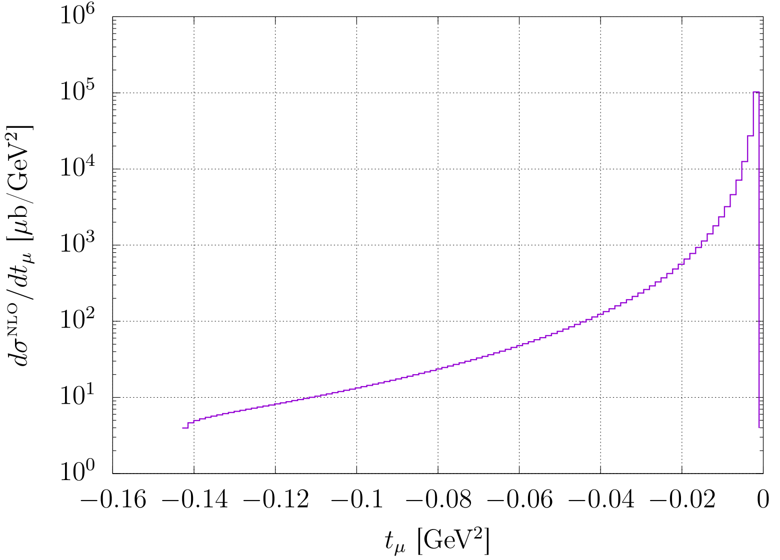
The Montecarlo code:

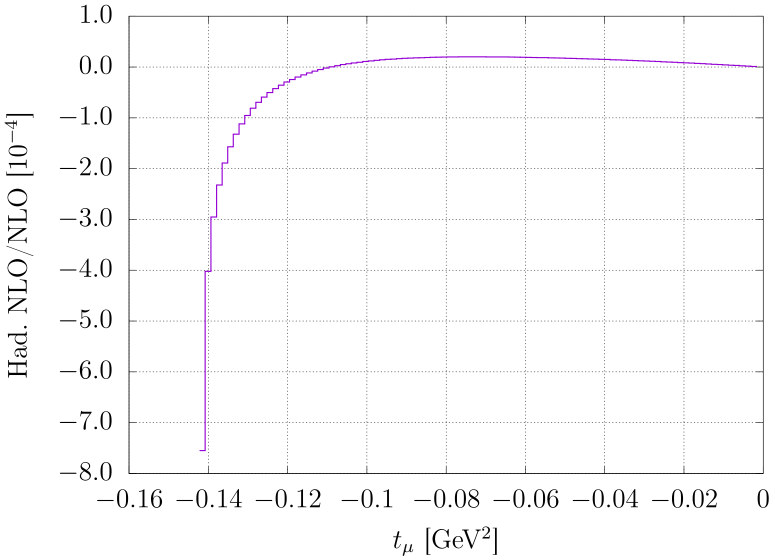
- $\Pi^{\text{had}}(t)$ and $R^{\text{had}}(z)$ provided by Jegerlehner's package alphaQED:
  www-com.physik.hu-berlin.de/∼fjeger/alphaQEDc17.tar.gz
- Good numerical stability with Collier when $z \gg s, t$.

$e^- \mu^+ \to e^- \mu^+$



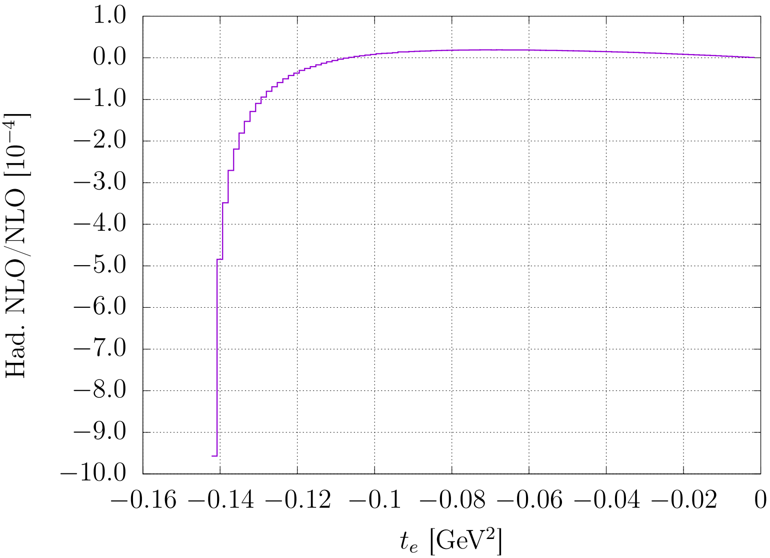
10^6
10^5
10^4
10^3
10^2
10^1
10^0
$d\sigma^{\rm NLO}/dt_\mu$ [μb/GeV^2]
−0.16 −0.14 −0.12 −0.1 −0.08 −0.06 −0.04 −0.02 0
t_μ [GeV^2]

$e^- \mu^+ \to e^- \mu^+$

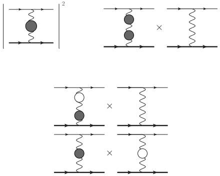


1.0
0.0
−1.0
−2.0
−3.0
−4.0
−5.0
−6.0
−7.0
−8.0
Had. NLO/NLO [10⁻⁴]
−0.16 −0.14 −0.12 −0.1 −0.08 −0.06 −0.04 −0.02 0
t_μ [GeV²]

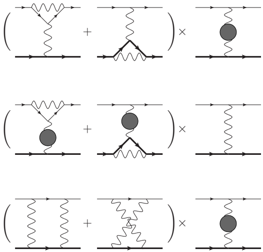
$e^- \mu^+ \to e^- \mu^+$


1.0
0.0
−1.0
−2.0
−3.0
−4.0
−5.0
−6.0
−7.0
−8.0
−9.0
−10.0
Had. NLO/NLO [10−4]
−0.16
−0.14
−0.12
−0.1
−0.08
−0.06
−0.04
−0.02
0
te [GeV2]

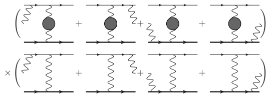
Class I



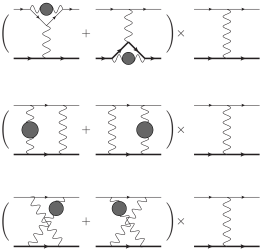
Class II



Class III



Class IV



step 1: $\left|\frac{\alpha(t)^2}{\alpha}\right| \frac{d\sigma^{\mathrm{LO}}}{dt} + \frac{d\sigma}{dt}^{\mathrm{NLO}} + \frac{d\sigma}{dt}^{\mathrm{QED\,NNLO}} \longrightarrow \Pi^{\mathrm{had}}_{(1)}(t)$

step 1: $\left|\frac{\alpha(t)^2}{\alpha}\right| \frac{d\sigma^{\mathrm{LO}}}{dt} + \frac{d\sigma}{dt}^{\mathrm{NLO}} + \frac{d\sigma}{dt}^{\mathrm{QED\,NNLO}} \longrightarrow \Pi^{\mathrm{had}}_{(1)}(t)$

step 2: $\left|\frac{\alpha(t)^2}{\alpha}\right| \frac{d\sigma^{\mathrm{LO}}}{dt} + \frac{d\sigma}{dt}^{\mathrm{NLO}} + \frac{d\sigma}{dt}^{\mathrm{QED\,NNLO}} + \frac{d\sigma}{dt}^{\mathrm{Had\,NLO}} \quad \Pi^{\mathrm{had}}_{(2)}(t)$

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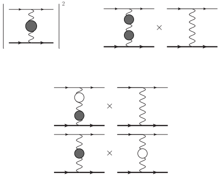
step 2: $\left|\frac{\alpha(t)^2}{\alpha}\right| \frac{d\sigma^{\mathrm{LO}}}{dt} + \frac{d\sigma}{dt}^{\mathrm{NLO}} + \frac{d\sigma}{dt}^{\mathrm{QED\,NNLO}} + \frac{d\sigma}{dt}^{\mathrm{Had\,NLO}} \longrightarrow \Pi^{\mathrm{had}}_{(2)}(t)$

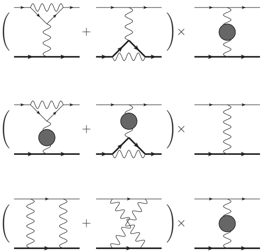
step 3: $\left|\frac{\alpha(t)^2}{\alpha}\right| \frac{d\sigma^{\mathrm{LO}}}{dt} + \frac{d\sigma}{dt}^{\mathrm{NLO}} + \frac{d\sigma}{dt}^{\mathrm{QED\,NNLO}} + \frac{d\sigma}{dt}^{\mathrm{Had\,NLO}} \longrightarrow \Pi^{\mathrm{had}}_{(3)}(t)$

...

Class I



Class II



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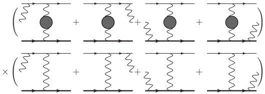

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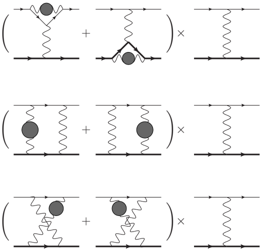

The Hyperspherical Approach

$$I = \int d^4k \, \frac{\Pi^{\mathrm{had}}(k^2)}{k^2} \cdots = -i \int_0^{+\infty} \frac{dk_E^2}{2} \, \Pi^{\mathrm{had}}(-k_E^2) \int d\Omega_4 \cdots$$

The Angular Integrals

Euclidean denominators expanded in Gegenbauer polynomials:

$$\frac{1}{(q-k)^2+m^2} = \frac{Z}{|q||k|}\sum_{n=0}^{\infty} Z^n C_n(\hat{q}\cdot\hat{k}),$$

with

$$Z = \frac{q^2+k^2+m^2-\lambda(q^2,k^2,-m^2)}{|q||k|}.$$

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The angular integrals:

$$\int \frac{d\Omega_4}{2\pi^2} C_n(\hat{a}\cdot\hat{k})C_m(\hat{k}\cdot\hat{b}) = \frac{\delta_{mn}}{n+1} C_n(\hat{a}\cdot\hat{b}),$$

$$C_n(x)C_m(x) = \sum_{j=0}^{\min(n,m)} C_{m+n-2j}(x).$$

Sixth order contributions to the electron g-2:

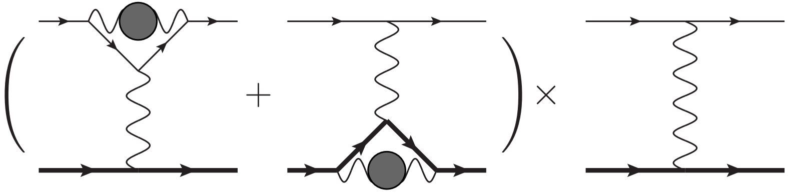
- Levine and Roskies, Phys. Rev. Lett. **30** (1973) 772.
- Levine and Roskies, Phys. Rev. D **9** (1974) 421.
- Levine, Remiddi and Roskies, Phys. Rev. D **20** (1979) 2068.
- Roskies, Levine and Remiddi, Adv. Ser. Direct. High Energy Phys. **7** (1990) 162.
- Laporta and Remiddi, Phys. Lett. B **265** (1991) 182.
- Laporta and Remiddi, Phys. Lett. B **301** (1993) 440.

Pion pole contribution to a_μ^{HLBL}:

- Knecht and Nyffeler, Phys. Rev. D **65** (2002) 073034.

Dispersive approach to a_μ^{HLBL}:

- Colangelo, Hoferichter, Procura and Stoffer, JHEP **1409**, 091 (2014).
- Colangelo, Hoferichter, Kubis, Procura and Stoffer, Phys. Lett. B **738** (2014) 6.
- Colangelo, Hoferichter, Procura and Stoffer, JHEP **1509** (2015) 074.

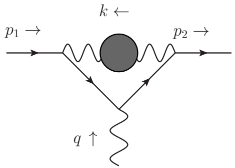


Vertex Correction with Hyperspherical Approach

$$\Gamma^{\mu}(q) = \gamma^{\mu} F_1(q^2) + i\frac{\sigma^{\mu\nu} q_{\nu}}{2m_{\ell}} F_2(q^2).$$

$$\Gamma^\mu(q) = \gamma^\mu F_1(q^2) + i\frac{\sigma^{\mu\nu} q_\nu}{2m_\ell} F_2(q^2).$$

Hadronic contribution to the lepton form factors:



$$F_{1,2}^{\rm had}(q^2) = \int \frac{d^4k}{(2\pi)^4}\left[-\Pi^{\rm had}(k^2)\right]\frac{\text{some numerator}}{D_0 D_1 D_2}$$

with

- $D_0 = k^2$;
- $D_{1,2} = (p_{1,2}+k)^2 - m_\ell^2$.

- Angular integrals:
  - $\int \frac{d\Omega_4}{2\pi^2} \frac{1}{D_i}$, with $i = 1, 2$;
  - $\int \frac{d\Omega_4}{2\pi^2} \frac{p_i \cdot k}{D_j}$, with $i \neq j$;
  - $\int \frac{d\Omega_4}{2\pi^2} \frac{1}{D_1 D_2}$.
- Analytic continuation $p_{E1}^2, p_{E2}^2 \to -m^2 - i\varepsilon$;
- Renormalization condition: $F_1^{\mathrm{had}}(0) = 0$;
- Change of variable: $k_E^2 \to m_\ell^2 x^2/(1-x)$.

Hadronic Contribution to the Form Factors

$$F_{1,2}^{\mathrm{had}}(q^2 = t) = \frac{\alpha}{\pi} \int_0^1 dx \Big[-\Pi^{\mathrm{had}} \left(\frac{m_\ell^2 x^2}{x-1} \right) \Big] f_{1,2}(t,x)$$

Hadronic Contribution to the Form Factors

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Check:

- a_μ^{HLO} for $t = 0$: $f_2(0, x) = 1 - x$.
  T. Blum, PRL 91 (2003) 052001

- Setting $\Pi^{\rm had}(q^2) = -1$ and integrating $f_2(t,x)$:

$$F_2^{(1l)}(q^2) = \frac{\alpha}{\pi} \frac{\xi \log \xi}{\xi^2 - 1}, \quad \text{with } q^2/m^2 = -(1-\xi)^2/\xi.$$

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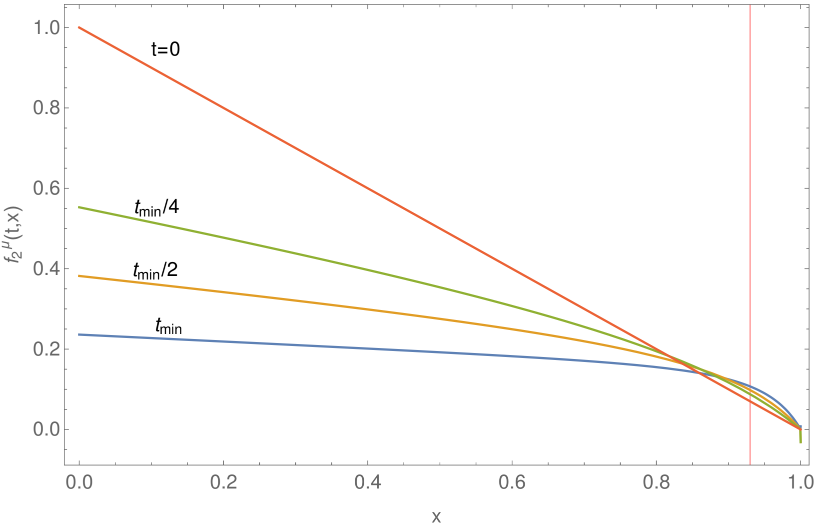
- Setting $\Pi^{\rm had}(q^2) = -q^2/(q^2-\lambda^2)$, integrating $f_1(t,x)$ and keeping non-vanishing terms in the limit $\lambda \to 0$:

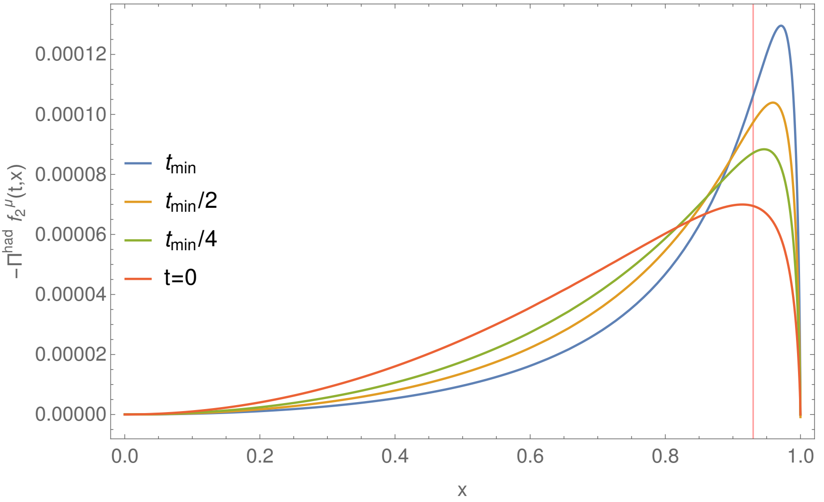
$$F_1^{(1l)}(q^2)\left(\frac{\alpha}{\pi}\right)^{-1} = \log\left(\frac{\lambda}{m}\right)\left[\frac{\xi^2+1}{\xi^2-1}\log(\xi) - 1\right] + \frac{3\xi^2+2\xi+3}{4\,(\xi^2-1)}\log(\xi) - 1 + \frac{1+\xi^2}{1-\xi^2}\left[\mathrm{Li}_2(-\xi) - \frac{\log^2(\xi)}{4} + \frac{\pi^2}{12} + \log(\xi)\log(\xi+1)\right].$$

Barbieri, Mignaco and Remiddi, Nuovo Cim. A **11** (1972) 824.

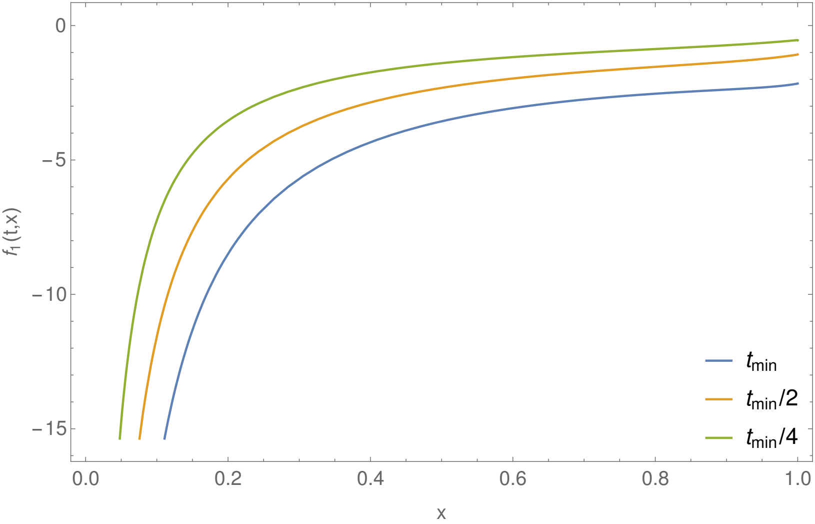
$f_2(t,x)$ with $\ell = \mu$



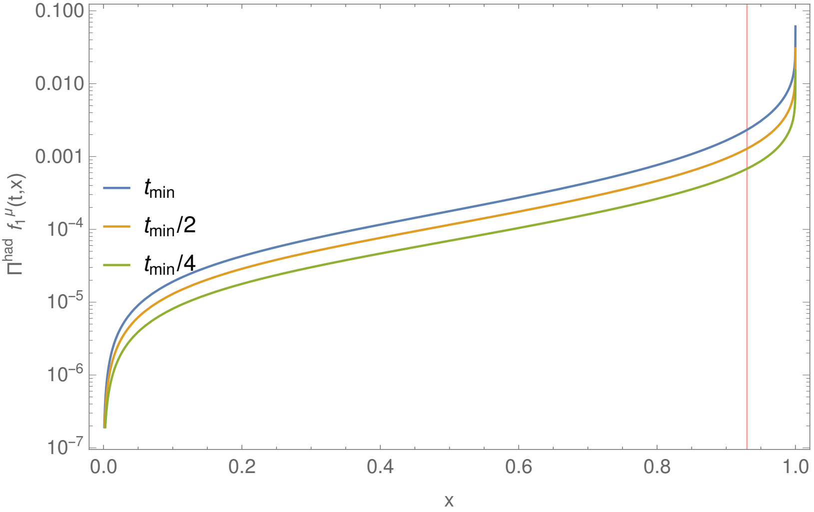
$-\Pi^{\text{had}}\, f_2(t,x)$ with $\ell = \mu$



$f_1(t,x)$ with $\ell = \mu$



$\Pi^{\text{had}} f_1(t,x)$ with $\ell = \mu$



Successful angular integration with Gegenbauer polynomials requires:

- planar graphs.
- the diagram must not be a box or contain box subgraph.

HEP **1** records found Search tool

1. **Hyperspherical integration and the triple cross vertex graphs**

S. Laporta (Bologna U. & INFN, Bologna). Feb 1994. 13 pp.
Published in **Nuovo Cim. A107 (1994) 1729-1738**
DFUB-94-01
DOI: 10.1007/BF02780705
e-Print: **hep-ph/9404203** | **PDF**

References | BibTeX | LaTeX(US) | LaTeX(EU) | Harvmac | EndNote
ADS Abstract Service

Detailed record - Cited by 2 records

Brute-force integration:

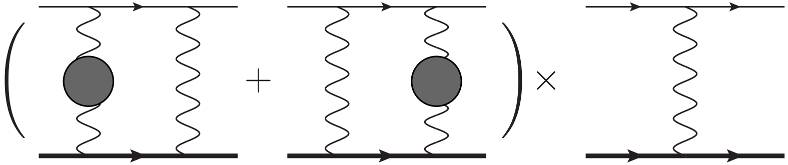
$$\int d\Omega_4 = \int_0^{\pi} \sin^2\theta_1 d\theta_1 \int_0^{\pi} \sin\theta_2 d\theta_2 \int_0^{2\pi} d\theta_3$$

- General two propagator case:

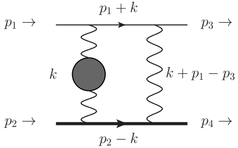
$$\int \frac{d\Omega_4}{[(k-p_1)^2+m_1^2]\,[(k-p_2)^2+m_2^2]}$$

- General three propagator case:

$$\int \frac{d\Omega_4}{[(k-p_1)^2+m_1^2]\,[(k-p_2)^2+m_2^2]\,[(k-p_3)^2+m_3^2]}$$



Boxes with Hyperspherical Approach



$$\mathrm{box}(s,t) = \int \frac{d^4k}{(2\pi)^4}\left[-\Pi^{\mathrm{had}}(k^2)\right]\frac{\text{some numerator}}{D_0 D_1 D_2 D_3}$$

$$D_0 = k^2, \qquad D_1 = (k+p_1)^2 - m_e^2,$$
$$D_2 = (k+p_1-p_3)^2, \quad D_3 = (k-p_2)^2 - m_\mu^2.$$

Tensor integrals *à la Passarino-Veltman*:

$$D^{\Pi}_{0,\mu,\mu\nu} \equiv \frac{1}{i\pi^2}\int d^4k\, \Pi^{\mathrm{had}}(k^2)\,\frac{[1, k_\mu, k_\mu k_\nu]}{D_0 D_1 D_2 D_3}$$

and their tensor coefficient functions: $D^{\Pi}_0, D^{\Pi}_i, D^{\Pi}_{00}, D^{\Pi}_{ij}$.

Boxes with Hyperspherical Approach

Tensor integral reduced to *angular* master integrals
(shift of loop momentum k not allowed):

- $\dfrac{1}{D_1 D_2}$,
- $\dfrac{1}{D_1 D_3}$,
- $\dfrac{1}{D_2 D_3}$,
- $\dfrac{1}{D_1 D_2 D_3}$,
- $\dfrac{p_1 \cdot k}{D_2 D_3}$,
- $\dfrac{p_2 \cdot k}{D_1 D_3}$,
- $\dfrac{(p_1 - p_3) \cdot k}{D_1 D_2}$.

Boxes with Hyperspherical Approach

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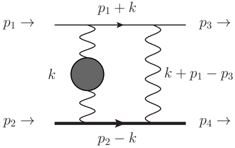
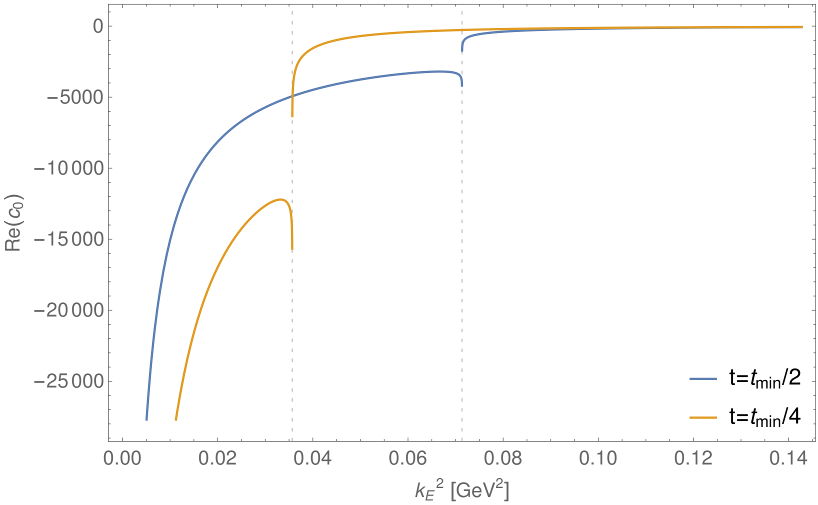
- $\dfrac{p_2 \cdot k}{D_1 D_3}$,
- $\dfrac{(p_1 - p_3) \cdot k}{D_1 D_2}$.

$$C_0^{\Pi}(m_e^2, m_e^2, t; 0, m_e^2, 0) = \frac{1}{i\pi^2} \int d^4k \, \frac{\Pi^{\rm had}(k^2)}{D_0 D_1 D_2}$$

$$= \int_0^{+\infty} dk_E^2 \, \Pi^{\rm had}(-k_E^2)\, c_0(t, k_E^2, m_e^2)$$

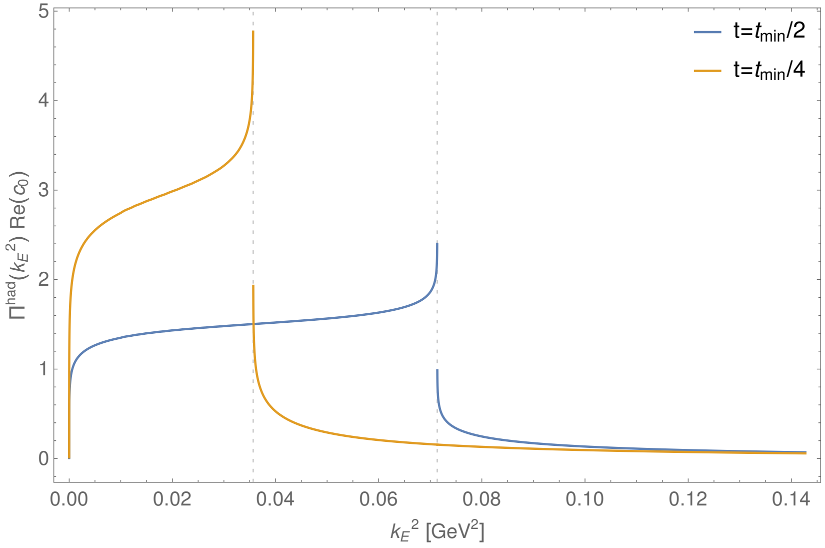


$c_0(t, k_E^2, m_e^2)$



0
−5000
−10 000
−15 000
−20 000
−25 000
Re(c_0)
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
$k_E{}^2$ [GeV^2]
t=$t_{\rm min}$/2
t=$t_{\rm min}$/4

$\Pi^{\text{had}}(-k_E^2)\, c_0(t, k_E^2, m_e^2)$



Hyperspherical vs Dispersive Approach

Dispersive approach

$$d = -\frac{1}{\pi} \int \frac{dz}{z} \, \mathrm{Im}\Pi^{\mathrm{lep}}(z) \, C_0(m_e^2, m_e^2, t, 0, m_e^2, z)$$

Hyperspherical approach

$$h = \int_0^{+\infty} dk_E^2 \, \Pi^{\mathrm{lep}}(-k_E^2) \, c_0(t, k_E^2, m_e^2)$$

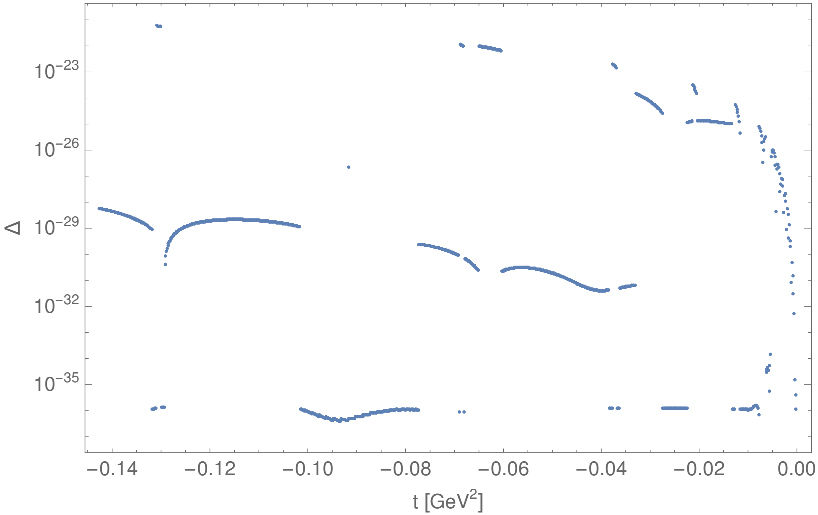
Numerical setup

- Numerical integration with `Mathematica`.
- One-loop scalar functions provided by `Package X`.

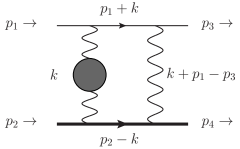
  H. Patel, Comput. Phys. Commun. 197, 276 (2015)
- The discrepancy:

$$\Delta = \frac{|d-h|}{|d+h|}.$$

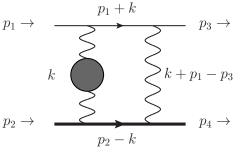
Hyperspherical vs Dispersive Approach



$$
\begin{aligned}
D_0^{\Pi}(m_e^2, m_e^2, m_\mu^2, m_\mu^2; t, s; 0, m_e^2, 0, m_\mu^2) \\
&= \frac{1}{i\pi^2} \int d^4k \, \frac{\Pi^{\mathrm{had}}(k^2)}{D_0 D_1 D_2 D_3} \\
&= \int_0^{+\infty} dk_E^2 \, \Pi^{\mathrm{had}}(-k_E^2) \, d_0(s, t, k_E^2, m_e^2, m_\mu^2)
\end{aligned}
$$



$$D_0^{\Pi}(m_e^2, m_e^2, m_\mu^2, m_\mu^2; t, s; 0, m_e^2, 0, m_\mu^2)$$
$$= \frac{1}{i\pi^2} \int d^4k \, \frac{\Pi^{\mathrm{had}}(k^2)}{D_0 D_1 D_2 D_3}$$
$$= \int_0^{+\infty} dk_E^2 \, \Pi^{\mathrm{had}}(-k_E^2) \, d_0(s, t, k_E^2, m_e^2, m_\mu^2)$$



IR singularity for $k_E^2 \to -t$

IR soft divergence

Subtraction Method:

$$\int_0^{+\infty} dk_E^2 \, \Pi^{\rm had}(-k_E^2) \, d_0(s, t, k_E^2, m_e^2, m_\mu^2) \to$$

$$\int_0^{+\infty} dk_E^2 \, \left[\Pi^{\rm had}(-k_E^2) - \Pi^{\rm had}(t) \frac{2k_E^2}{k_E^2 - t}\right] \, d_0(s, t, k_E^2, m_e^2, m_\mu^2)$$

$$+ 2\Pi^{\rm had}(t) \int_0^{+\infty} dk_E^2 \, \frac{k_E^2}{k_E^2 - t} d_0(s, t, k_E^2, m_e^2, m_\mu^2)$$

IR soft divergence

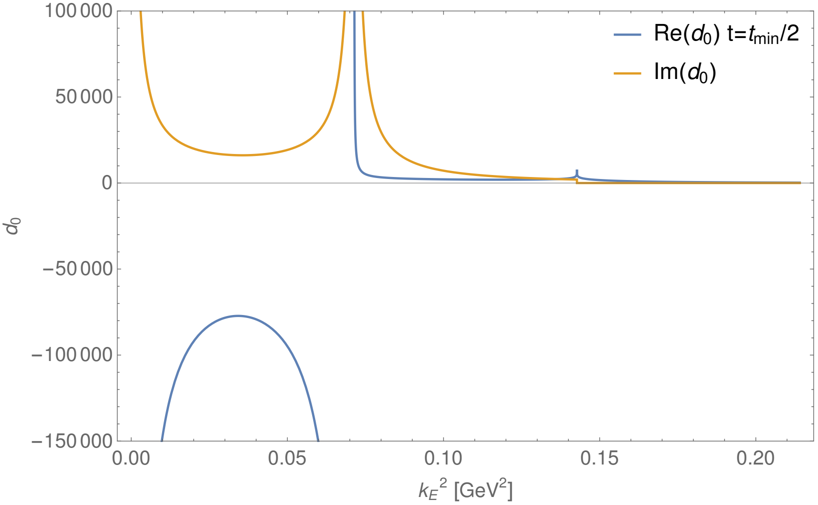
Subtraction Method:

$$\int_0^{+\infty} dk_E^2\, \Pi^{\mathrm{had}}(-k_E^2)\, d_0(s,t,k_E^2,m_e^2,m_\mu^2) \to$$

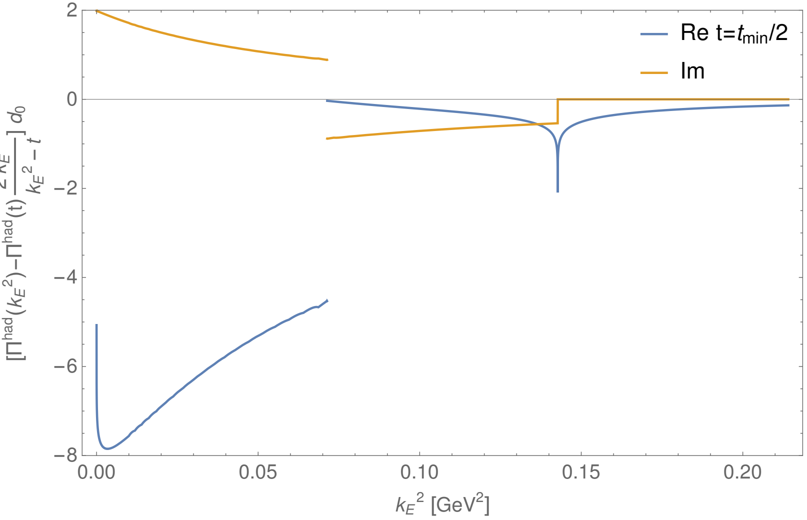
$$\int_0^{+\infty} dk_E^2\, \left[\Pi^{\mathrm{had}}(-k_E^2) - \Pi^{\mathrm{had}}(t)\frac{2k_E^2}{k_E^2 - t}\right]\, d_0(s,t,k_E^2,m_e^2,m_\mu^2)$$

$$+\, 2\Pi^{\mathrm{had}}(t) \int \frac{dk^4}{i\pi^2} \frac{1}{(k^2 - |t|)D_1 D_2 D_3}$$

$d_0(s, t, k_E^2, m_e^2, m_\mu^2)$



$$\left[\Pi^{\rm had}(-k_E^2) - \Pi^{\rm had}(t)\frac{2k_E^2}{k_E^2 - t}\right]\ d_0(s, t, k_E^2, m_e^2, m_\mu^2)$$



Hyperspherical vs Dispersive Approach

Dispersive approach

$$d = -\frac{1}{\pi} \int \frac{dz}{z} \, \mathrm{Im}\Pi^{\mathrm{lep}}(z) \, D_0(m_e^2, m_e^2, m_\mu^2, m_\mu^2; t, s; 0, m_e^2, z, m_\mu^2)$$

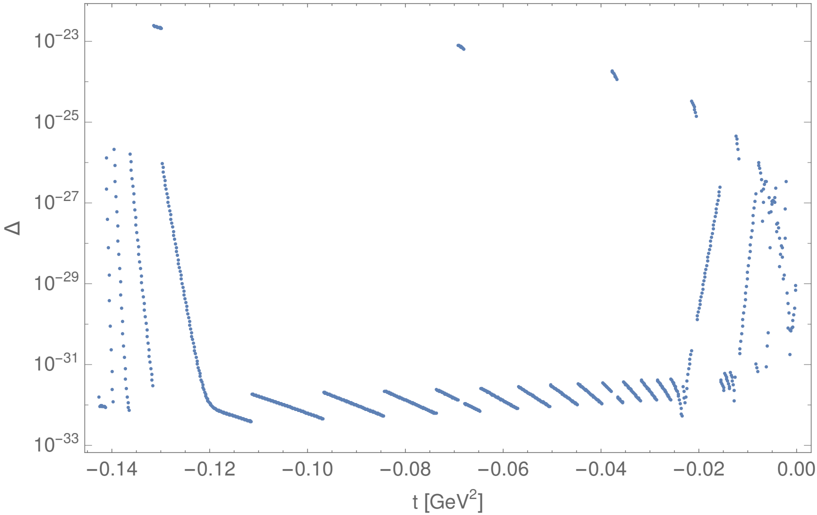
Hyperspherical approach

$$h = \int_0^{+\infty} \left[\Pi^{\mathrm{lep}}(-k_E^2) - \Pi^{\mathrm{lep}}(t) \frac{2k_E^2}{k_E^2 - t} \right] d_0(s, t, k_E^2, m_e^2, m_\mu^2) \, d k_E^2$$
$$+ 2 \, \Pi^{\mathrm{had}}(t) \, D_0(m_e^2, m_e^2, m_\mu^2, m_\mu^2; t, s; 0, m_e^2, |t|, m_\mu^2)$$

- Compare the discrepancy:

$$\Delta = \frac{|d - h|}{|d + h|}.$$

Hypersperical vs Dispersive Approach



To-do List & Conclusions

- First Montecarlo for Had. NLO corrections with dispersive approach.
- Class IV diagrams can be evaluated with space-like data via hyperspherical approach.
- Class IV vertex corrections ✓
- Class IV box diagrams . . .

To-do List & Conclusions

- Compare Class IV with $\Pi^{\mathrm{had}}(q^2) \to \Pi^{\mathrm{lep}}(q^2)$ with exact QED two-loop results.
- Produce a numerically stable implementation.

- Can $\Pi^{\mathrm{had}}(t)$ be measured in an iterative way?
- Why don't we calculate also $a_\mu^{HNLO-HNNLO}$ with space-like data?