A local analytic sector subtraction at NNLO

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MITP, 08/2018

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Motivation

- Various recipes already available for NNLO subtraction/slicing.
  - Slicing: simpler but approximate, need to check cutoff independence.
    qT [Catani, Grazzini, et al.], N-jettiness [Boughezal, Petriello, et al.], [Gaunt, Tackmann, et al.].
  - Subtraction: more complex but exact.
    Antennae [Gehrmann, Glover, et al.], sector-improved [Czakon, Mitov, et al.], nested soft-collinear [Caola, Melnikov, et al.], colourful [Del Duca, Troscanyi, et al.], projection to Born [Salam, et al.], sector decomposition [Anastasiou, et al.], [Binoth, et al.], $\mathcal{E}$-prescription [Frixione, Grazzini], geometric [Herzog].
  - Unsubtraction: loop-tree duality [Rodrigo, et al.].
- Complexity in the structure of the subtraction scheme increases substantially with respect to NLO, NNLO subtraction problem not yet fully solved in general, room for studies.

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- Complexity in the structure of the subtraction scheme increases substantially with respect to NLO, NNLO subtraction problem not yet fully solved in general, room for studies.
- Our main motivation for studying a new scheme:
  - investigate how much one can simplify subtraction and involved calculations;
  - understand what properties/choices of NLO subtraction terms can be usefully exported to NNLO.
- In the following, still partial results on massless and final-state-only QCD partons.

NLO

Subtracted NLO cross sections

- NLO coefficient of the differential cross section (X = IRC safe, X_i = observable computed with i-body kinematics, $\delta_i \equiv \delta(X - X_i)$):

$$\frac{d\sigma_{\rm NLO}}{dX} = \int d\Phi_n \, V \, \delta_n + \int d\Phi_{n+1} \, R \, \delta_{n+1} .$$

- Add and subtract counterterm:

$$\int d\widehat{\Phi}_{n+1} \, \overline{K} \, \delta_n$$

- $\overline{K}$ = local counterterm: same phase-space singularities as R, but simple enough to be integrated analytically in d dimensions.

- d-dimensional integrated counterterm:

$$I = \int d\widehat{\Phi}_{\rm rad} \, \overline{K} \, , \qquad d\widehat{\Phi}_{\rm rad} = d\widehat{\Phi}_{n+1} / d\Phi_n .$$

- Subtracted NLO coefficient

$$\frac{d\sigma_{\rm NLO}}{dX} = \int d\Phi_n \, (V + I) \, \delta_n + \int \left(d\Phi_{n+1} \, R \, \delta_{n+1} - d\widehat{\Phi}_{n+1} \, \overline{K} \, \delta_n \right).$$

- Both integrals $\int (V + I)$ and $\int (R - \overline{K})$ are separately finite and evaluated numerically in $d = 4$.

NLO sectors (à la FKS) [Frixione, Kunszt, Signer]

- Partition phase space Φ_{n+1} with sector functions $\mathcal{W}_{ij}$, (normalised as $\sum_{i,\, j\neq i} \mathcal{W}_{ij} = 1$), such that $R\,\mathcal{W}_{ij}$ is singular only in one soft ($\mathbf{S}_i$) and one collinear ($\mathbf{C}_{ij}$) configuration.
- Properties:

$$\mathbf{S}_i \sum_{k\neq i} \mathcal{W}_{ik} = 1\,, \qquad \mathbf{C}_{ij} \sum_{ab\,\in\,\mathrm{perm}(ij)} \mathcal{W}_{ab} = 1\,, \qquad \longleftarrow \quad \text{sum rules}$$

- Sum rules: by summing over all sectors sharing the same singularity the $\mathcal{W}$'s disappear. Key for simplifying analytic integration of $\overline{K}$.
- Example of sector functions ($s_{qi} = 2\, q_{\mathrm{cm}} \cdot k_i, \quad s_{ij} = 2\, k_i \cdot k_j$):

$$\mathcal{W}_{ij} = \frac{\sigma_{ij}}{\sum\limits_{k,\, l\neq k} \sigma_{kl}}\,, \qquad \text{with} \qquad \sigma_{ij} = \frac{1}{e_i\, w_{ij}}\,, \qquad e_i = \frac{s_{qi}}{s}\,, \qquad w_{ij} = \frac{s\, s_{ij}}{s_{qi}\, s_{qj}}\,.$$

Structure of NLO subtraction

- Singularity of real matrix element in sector ij known in advance in terms of dot products s_{ab}, without parametrising the sector (at variance with FKS).

- $\mathbf{S}_i\, R \;=\;$ soft limit for parton i = leading term in R as $k_i^\mu \to 0$.
  $\mathbf{C}_{ij}\, R \;=\;$ collinear limit for partons ij = leading term in R as relative $k_\perp^\mu \to 0$.

$$\mathbf{S}_i\, R(\{k\}) \;=\; -\mathcal{N}_1 \sum_{l,\,m} \delta_{f_i g}\, \frac{s_{lm}}{s_{il}\, s_{im}}\, B_{lm}(\{k\}_{\not i})\;,$$

$$\mathbf{C}_{ij}\, R(\{k\}) \;=\; \frac{\mathcal{N}_1}{s_{ij}} \Big[P_{ij}\, B\Big(\{k\}_{\not i \not j}, k\Big) + Q_{ij}^{\mu\nu}\, B_{\mu\nu}\Big(\{k\}_{\not i \not j}, k\Big) \Big]\,,$$

$$\mathbf{S}_i\, \mathbf{C}_{ij}\, R(\{k\}) \;=\; 2\,\mathcal{N}_1\, C_{f_j}\, \delta_{f_i g}\, \frac{s_{jr}}{s_{ij}\, s_{ir}}\, B\big(\{k\}_{\not i}\big)\;.$$

- Argument of Altarelli-Parisi kernel P_{ij} is $x_i = s_{ir}/(s_{ir}+s_{jr})$, with $r \neq i, j$.

- Candidate counterterm in sector ij: $\quad K_{ij} \;=\; (\mathbf{S}_i + \mathbf{C}_{ij} - \mathbf{S}_i\, \mathbf{C}_{ij}) R\, \mathcal{W}_{ij}$
  (limits applied to both R and $\mathcal{W}_{ij}$).

- As minimal as FKS, but no parametrisation yet: freedom to be exploited to simplify analytic integration.

Mapping from NLO to Born kinematics (à la CS) [Catani, Seymour]

- Need a momentum mapping $\{k_1, ..., k_{n+1}\} \to \{\bar{k}_1, ..., \bar{k}_n\}$ to factorise radiation phase space from Born phase-space, and integrate conuterterm in the latter.
- Catani-Seymour final-state massless mapping $\{k\} \to \{\bar{k}\}^{(abc)}$:

$$\bar{k}_i^{(abc)} = k_i, \quad \text{if } i \neq a, b, c,$$

$$\bar{k}_b^{(abc)} = k_a + k_b - \frac{s_{ab}}{s_{ac} + s_{bc}}\, k_c\,, \qquad \bar{k}_c^{(abc)} = \frac{s_{abc}}{s_{ac} + s_{bc}}\, k_c\,,$$

with $s_{abc} = s_{ab} + s_{ac} + s_{bc}$, and $\bar{k}_b^{(abc)} + \bar{k}_c^{(abc)} = k_a + k_b + k_c$.

Local-counterterm definition

- Mapping $\{k\} \to \{\bar{k}\}^{(abc)}$: at this stage there is still full freedom to choose labels a, b, c as we want. Adapt the choice to the invariants appearing in the kernels.

- $\mathbf{C}_{ij}\, R$ features invariants s_{ij}, s_{ir}, and s_{jr}: choose $(abc) = (ijr)$.

- Each term in the eikonal sum in $\mathbf{S}_i\, R$ features s_{il}, s_{im}, and s_{lm}: choose $(abc) = (ilm)$.

- Remapped singular limits:

$$\begin{aligned}
\overline{\mathbf{S}}_i\, R(\{k\}) &= -\mathcal{N}_1 \sum_{l,\,m} \delta_{f_i g} \frac{s_{lm}}{s_{il}\, s_{im}} B_{lm}\Big(\{\bar{k}\}^{(ilm)}\Big)\,, \\
\overline{\mathbf{C}}_{ij}\, R(k) &= \frac{\mathcal{N}_1}{s_{ij}} \Big[P_{ij}\, B\Big(\{\bar{k}\}^{(ijr)}\Big) + Q_{ij}^{\mu\nu}\, B_{\mu\nu}\Big(\{\bar{k}\}^{(ijr)}\Big)\Big]\,, \\
\overline{\mathbf{S}}_i\, \overline{\mathbf{C}}_{ij}\, R(\{k\}) &= 2\,\mathcal{N}_1\, C_{f_j}\, \delta_{f_i g} \frac{s_{jr}}{s_{ij}\, s_{ir}} B\Big(\{\bar{k}\}^{(ijr)}\Big)\,,
\end{aligned}$$

- Local-counterterm definition:

$$\overline{K} = \sum_{i,j\neq i} \overline{K}_{ij}\,, \qquad \overline{K}_{ij} \equiv \big(\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\, \overline{\mathbf{C}}_{ij}\big)\, R\, \mathcal{W}_{ij}\,,$$

where barred limits on $\mathcal{W}$'s act as unbarred ones.

NLO-counterterm integration (I)

- Catani-Seymour variables $y, z \in [0,1]$ for mapping $\{k\} \to \{\bar{k}\}^{(abc)}$:

$$s_{ab} = y\, s_{abc}\,, \qquad s_{ac} = z(1-y)\, s_{abc}\,, \qquad s_{bc} = (1-z)(1-y)\, s_{abc}\,.$$

- Phase-space factorisation:

$$d\Phi_{n+1} = d\Phi_n^{(abc)}\, d\Phi_{\rm rad}^{(abc)}\,, \qquad d\Phi_{\rm rad}^{(abc)} \equiv d\Phi_{\rm rad}\left(\bar{s}_{bc}^{(abc)}; y, z, \phi\right)\,,$$

$$\int d\Phi_{\rm rad}\,(s; y, z, \phi) \equiv N(\epsilon)\, s^{1-\epsilon} \int_0^\pi d\phi\, \sin^{-2\epsilon}\phi \int_0^1 dy \int_0^1 dz \Big[y(1-y)^2\, z(1-z)\Big]^{-\epsilon} (1-y)\,,$$

$$N(\epsilon) \equiv \frac{(4\pi)^{\epsilon-2}}{\sqrt{\pi}\,\Gamma(1/2-\epsilon)}\,, \qquad \bar{s}_{bc}^{(abc)} \equiv 2\,\bar{k}_b^{(abc)} \cdot \bar{k}_c^{(abc)} = s_{abc}\,.$$

- ϕ = azimuth between $\vec{k}_a$ and an reference three-momentum ($\neq \vec{k}_b, \vec{k}_c$).

NLO-counterterm integration (II)

$$
\begin{aligned}
\overline{K} &= \sum_{i,j\neq i} \overline{K}_{ij} = \sum_{i,j\neq i} \left[\,\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij}(1-\overline{\mathbf{S}}_i)\right] R\,\mathcal{W}_{ij} \\
&= \sum_i (\overline{\mathbf{S}}_i R)\left[\mathbf{S}_i \sum_{j\neq i} \mathcal{W}_{ij}\right] + \sum_{i,j>i} (\overline{\mathbf{C}}_{ij} R)\left[\mathbf{C}_{ij}(\mathcal{W}_{ij}+\mathcal{W}_{ji})\right] - \sum_{i,j\neq i} \overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij} R \\
&= \sum_i \overline{\mathbf{S}}_i\,R + \sum_{i,\,j>i} \overline{\mathbf{C}}_{ij}\,\left(1 - \overline{\mathbf{S}}_i - \overline{\mathbf{S}}_j\right) R\,. \qquad \longleftarrow \quad \text{used sum rules}
\end{aligned}
$$

- Integrated counterterm, explicit soft example (ς_k = symmetry factor of k-body ph. sp.):

$$
\begin{aligned}
I^{\,\mathrm{s}} &= -\mathcal{N}_1 \frac{\varsigma_{n+1}}{\varsigma_n} \sum_i \sum_{\substack{l\neq i\\ m\neq i}} B_{lm}\left(\{\bar{k}\}^{(ilm)}\right) \int d\Phi_{\mathrm{rad}}^{(ilm)}\, \delta_{f_i g}\, \frac{s_{lm}}{s_{li}s_{mi}} \\
&= -\mathcal{N}_1 \frac{\varsigma_{n+1}}{\varsigma_n} \sum_i \delta_{f_i g} \sum_{\substack{l\neq i\\ m\neq i}} B_{lm}\left(\{\bar{k}\}^{(ilm)}\right) \frac{1}{\bar{s}_{lm}^{(ilm)}} \int d\Phi_{\mathrm{rad}}\left(\bar{s}_{lm}^{(ilm)}; y, z, \phi\right) \frac{1-z}{yz}
\end{aligned}
$$

- Measure and integrand completely factorised, results available at all orders in ϵ:

$$
I^{\,\mathrm{s}} = -\mathcal{N}_1 \frac{\varsigma_{n+1}}{\varsigma_n} \sum_i \delta_{f_i g} \sum_{\substack{l\neq i\\ m\neq i}} B_{lm}\left(\{\bar{k}\}^{(ilm)}\right) \frac{(4\pi)^{\epsilon-2}}{\left(\bar{s}_{lm}^{(ilm)}\right)^{\epsilon}} \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2\,\Gamma(2-3\epsilon)}\,.
$$

NLO-counterterm integration (III)

- Full result, including hard-collinear (note: $\int d\Phi_{\rm rad}\, Q^{\mu\nu}_{ij} = 0$):

$$
\begin{aligned}
I(\{\bar k\}) \quad = \quad & -\mathcal{N}_1 \sum_p \frac{(4\pi)^{\epsilon-2}}{\bar s_{pr}^{\,\epsilon}} \, \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon\,\Gamma(2-3\epsilon)} \, \mathbb{C}\, B\big(\{\bar k\}\big) \\
& -\mathcal{N}_1 \sum_{l,\, m\neq l} \frac{(4\pi)^{\epsilon-2}}{\bar s_{lm}^{\,\epsilon}} \, \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2\,\Gamma(2-3\epsilon)} \, B_{lm}\big(\{\bar k\}\big)\,,
\end{aligned}
$$

  with $\mathbb{C} = \frac{C_A + 4\, T_R\, N_f}{2(3-2\epsilon)}\, \delta_{f_p g} + \frac{C_F}{2}\delta_{f_p\{q,\bar q\}}$.

- Used $d\widehat{\Phi}_{\rm rad} = d\Phi_{\rm rad}$ in the massless final-state case. This may be reconsidered in the general case.

- Virtual ϵ poles analytically reproduced in general.
  Finite parts checked differentially in a variety of cases.

NLO summary

- Sectors are useful for minimising the subtraction problem, as pioneered by FKS.
- Enhanced flexibility from parametrisation-independent (i.e. in terms of dot products) counterterm definition and from adaption of parametrisation/mapping to the involved invariants $\implies$ simplifications in analytic counterterm integration.
- Subtraction scheme at NLO like a bridge between FKS and CS, retaining the strengths of both (sector approach, and minimal structure from FKS; Lorentz invariance, and phase-space mappings/parametrisations from Catani-Seymour).
- These features can be exported to NNLO.

NNLO

Subtracted NNLO cross sections (I)

- NNLO coefficient of the differential cross section:

$$\frac{d\sigma_{\rm NNLO}}{dX} = \int d\Phi_n \, VV \, \delta_n + \int d\Phi_{n+1} \, RV \, \delta_{n+1} + \int d\Phi_{n+2} \, RR \, \delta_{n+2}.$$

- Add and subtract

$$\int d\widehat{\Phi}_{n+2} \, \overline{K}^{\,(\mathbf{1})} \, \delta_{n+1} \, , \qquad \int d\widehat{\Phi}_{n+2} \, (\overline{K}^{\,(\mathbf{2})} + \overline{K}^{\,(\mathbf{12})}) \, \delta_n \, , \qquad \int d\widehat{\Phi}_{n+1} \, \overline{K}^{(\mathbf{RV})} \, \delta_n \, .$$

- $\overline{K}^{\,(\mathbf{1})}$ and $(\overline{K}^{\,(\mathbf{2})} + \overline{K}^{\,(\mathbf{12})})$: same single- and double-unresolved singularities as RR.
  $\overline{K}^{\,(\mathbf{2})} \to$ double-unresolved limits (dubbed pure);
  $\overline{K}^{\,(\mathbf{12})} \to$ single-unresolved limits of double-unresolved ones (dubbed mixed);
  $\overline{K}^{(\mathbf{RV})} \to$ same phase-space singularities as RV.

- d-dimensional integrated counterterms:

$$I^{\,(\mathbf{1})} = \int d\widehat{\Phi}_{\rm rad,1} \, \overline{K}^{\,(\mathbf{1})} \, , \qquad I^{\,(\mathbf{2})} = \int d\widehat{\Phi}_{\rm rad,2} \, \overline{K}^{\,(\mathbf{2})} \, ,$$

$$I^{\,(\mathbf{12})} = \int d\widehat{\Phi}_{\rm rad,1} \, \overline{K}^{\,(\mathbf{12})} \, , \qquad I^{(\mathbf{RV})} = \int d\widehat{\Phi}_{\rm rad} \, \overline{K}^{(\mathbf{RV})} \, ,$$

where $d\widehat{\Phi}_{\rm rad,1} = d\widehat{\Phi}_{n+2}/d\widehat{\Phi}_{n+1}$, $d\widehat{\Phi}_{\rm rad,2} = d\widehat{\Phi}_{n+2}/d\Phi_n$, and $d\widehat{\Phi}_{\rm rad} = d\widehat{\Phi}_{n+1}/d\Phi_n$.

Subtracted NNLO cross sections (II)

- Subtracted NNLO coefficient of the differential cross section:

$$\frac{d\sigma_{\rm NNLO}}{dX} = \int d\Phi_n \left(VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})}\right) \delta_n$$
$$+ \int \left[\left(d\Phi_{n+1}\, RV + d\widehat{\Phi}_{n+1}\, I^{(\mathbf{1})}\right) \delta_{n+1} - d\widehat{\Phi}_{n+1} \left(\overline{K}^{(\mathbf{RV})} - I^{(\mathbf{12})}\right) \delta_n \right]$$
$$+ \int \left[d\Phi_{n+2}\, RR\, \delta_{n+2} - d\widehat{\Phi}_{n+2}\, \overline{K}^{(\mathbf{1})}\, \delta_{n+1} - d\widehat{\Phi}_{n+2} \left(\overline{K}^{(\mathbf{2})} + \overline{K}^{(\mathbf{12})}\right) \delta_n \right].$$

- Singularity-cancellation pattern:
  - $RR - \overline{K}^{(\mathbf{1})} - (\overline{K}^{(\mathbf{2})} + \overline{K}^{(\mathbf{12})})$ finite in $d = 4$, and in Φ_{n+2}.
  - $RV + I^{(\mathbf{1})}$ finite in $d = 4$, but singular in Φ_{n+1}.
  - $\overline{K}^{(\mathbf{RV})} - I^{(\mathbf{12})}$ finite in $d = 4$, but singular in Φ_{n+1}.
  - $RV + I^{(\mathbf{1})} - (\overline{K}^{(\mathbf{RV})} - I^{(\mathbf{12})})$ finite in $d = 4$, and in Φ_{n+1}.
  - $VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})}$ finite in $d = 4$, and in Φ_n.

NNLO sectors

- Introduce a partition of Φ_{n+2} through sector functions $\mathcal{W}_{ijkl}$, normalised as $\sum_{ijkl} \mathcal{W}_{ijkl} = 1$, to select as few singularities at a time as possible.

- Indices ij linked to single-unresolved singularities, k and l may or may not be $= j$, but $k \neq l$. Three possible combinations: $\mathcal{W}_{ijjk}$, $\mathcal{W}_{ijkj}$, and $\mathcal{W}_{ijkl}$. We define:

$$\mathcal{W}_{ijkl} = \frac{\sigma_{ijkl}}{\sum\limits_{a,\, b\neq a} \sum\limits_{\substack{c\neq a \\ d\neq a,c}} \sigma_{abcd}} \,, \qquad \sigma_{ijkl} = \frac{1}{e_i^\alpha \, w_{ij}^\beta} \frac{1}{(e_k + \delta_{kj}\, e_i)\, w_{kl}} \,, \qquad \alpha > \beta > 1 \,.$$

- In each sector, $RR\,\mathcal{W}_{abcd}$ is singular only in few kinematic configurations

$$\begin{array}{lcl} \mathcal{W}_{ijjk} & : & \mathbf{S}_i \,, \ \ \mathbf{C}_{ij} \,, \ \ \mathbf{S}_{ij} \,, \ \ \mathbf{C}_{ijk} \,, \ \ \mathbf{SC}_{ijk} \,; \\ \mathcal{W}_{ijkj} & : & \mathbf{S}_i \,, \ \ \mathbf{C}_{ij} \,, \ \ \mathbf{S}_{ik} \,, \ \ \mathbf{C}_{ijk} \,, \ \ \mathbf{SC}_{ijk} \,, \ \ \mathbf{CS}_{ijk} \,; \\ \mathcal{W}_{ijkl} & : & \mathbf{S}_i \,, \ \ \mathbf{C}_{ij} \,, \ \ \mathbf{S}_{ik} \,, \ \ \mathbf{C}_{ijkl} \,, \ \ \mathbf{SC}_{ikl} \,, \ \ \mathbf{CS}_{ijk} \,. \end{array}$$

- $\mathbf{S}_{ab} : e_a,\, e_b \rightarrow 0 \,,\, e_a/e_b \rightarrow$ constant, and similarly for $\mathbf{C}_{ijk}$, $\mathbf{C}_{ijkl}$. $\mathbf{SC}_{iab} = \mathbf{C}_{ab}\mathbf{S}_i$, and similarly for $\mathbf{CS}_{ijk}$.

- All limits above commute when acting on RR and on $\mathcal{W}$'s.

NNLO sectors: properties (I)

- Sum rules in double-unresolved limits: by summing over all sectors sharing the same singularity, $\mathcal{W}$ functions disappear.

$$\mathbf{S}_{ik}\left(\sum_{b\neq i}\sum_{d\neq i,k}\mathcal{W}_{ibkd}+\sum_{b\neq k}\sum_{d\neq k,i}\mathcal{W}_{kbid}\right)=1\,,$$

$$\mathbf{C}_{ijk}\sum_{abc\,\in\,\mathrm{perm}(ijk)}\left(\mathcal{W}_{abbc}+\mathcal{W}_{abcb}\right)=1\,,\qquad \mathbf{C}_{ijkl}\sum_{\substack{ab\,\in\,\mathrm{perm}(ij)\\ cd\,\in\,\mathrm{perm}(kl)}}\left(\mathcal{W}_{abcd}+\mathcal{W}_{cdab}\right)=1\,,$$

$$\mathbf{SC}_{ikl}\sum_{b\neq i}\left(\mathcal{W}_{ibkl}+\mathcal{W}_{iblk}\right)=1\,,\qquad \mathbf{CS}_{ijk}\left(\sum_{d\neq i,k}\mathcal{W}_{ijkd}+\sum_{d\neq j,k}\mathcal{W}_{jikd}\right)=1\,.$$

- Key for simplifying analytic integration of double-unresolved counterterms.

NNLO sectors: properties (I)

- In the single-unresolved limits, NNLO sector functions factorise NLO sector functions ($[ij]$ = parent of i and j)

$$
\begin{aligned}
\mathbf{C}_{ij}\,\mathcal{W}_{ijjk} &= \mathcal{W}_{[ij]k}\,\mathbf{C}_{ij}\,\mathcal{W}_{ij}^{(\alpha\beta)}\,, & \mathbf{S}_{i}\,\mathcal{W}_{ijjk} &= \mathcal{W}_{jk}\,\mathbf{S}_{i}\,\mathcal{W}_{ij}^{(\alpha\beta)}\,,\\
\mathbf{C}_{ij}\,\mathcal{W}_{ijkj} &= \mathcal{W}_{k[ij]}\,\mathbf{C}_{ij}\,\mathcal{W}_{ij}^{(\alpha\beta)}\,, & \mathbf{S}_{i}\,\mathcal{W}_{ijkj} &= \mathcal{W}_{kj}\,\mathbf{S}_{i}\,\mathcal{W}_{ij}^{(\alpha\beta)}\,,\\
\mathbf{C}_{ij}\,\mathcal{W}_{ijkl} &= \mathcal{W}_{kl}\,\mathbf{C}_{ij}\,\mathcal{W}_{ij}^{(\alpha\beta)}\,, & \mathbf{S}_{i}\,\mathcal{W}_{ijkl} &= \mathcal{W}_{kl}\,\mathbf{S}_{i}\,\mathcal{W}_{ij}^{(\alpha\beta)}\,,
\end{aligned}
$$

where

$$
\mathcal{W}_{ij}^{(\alpha\beta)} = \frac{\sigma_{ij}^{(\alpha\beta)}}{\sum\limits_{a,\,b\neq a}\sigma_{ab}^{(\alpha\beta)}}\,, \qquad \sigma_{ab}^{(\alpha\beta)} = \frac{1}{(e_a)^{\alpha}(w_{ab})^{\beta}}\,.
$$

with the same properties of NLO sector functions.

- This factorisation property will be important to have full NNLO single-unresolved integrated poles in each NLO sector.

- This allows $(RV + I^{(\mathbf{1})})$ and $(K^{(\mathbf{RV})} - I^{(\mathbf{12})})$ to be finite in $d = 4$ NLO sector by NLO sector.

NNLO counterterms

- In each sector, candidate (i.e. not yet momentum-remapped) counterterms built collecting singular limits of $RR\,\mathcal{W}$, written in terms of dot products.

- Example for sector $\mathcal{W}_{ijkj}$ (where nonzero limits are $\mathbf{S}_i$, $\mathbf{C}_{ij}$, $\mathbf{S}_{ik}$, $\mathbf{C}_{ijk}$, $\mathbf{SC}_{ijk}$, $\mathbf{CS}_{ijk}$):

$$K^{(\mathbf{1})}_{ijkj} = \Big[\mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)\Big]\, RR\,\mathcal{W}_{ijkj},$$

$$K^{(\mathbf{2})}_{ijkj} = \Big[\mathbf{S}_{ik} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ik}) + \mathbf{SC}_{ijk}(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk}) + \mathbf{CS}_{ijk}(1-\mathbf{SC}_{ijk})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk})\Big]\, RR\,\mathcal{W}_{ijkj},$$

$$K^{(\mathbf{12})}_{ijkj} = -\Big[\mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)\Big]\Big[\mathbf{S}_{ik} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ik}) + \mathbf{SC}_{ijk}(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk}) + \mathbf{CS}_{ijk}(1-\mathbf{SC}_{ijk})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk})\Big]\, RR\,\mathcal{W}_{ijkj},$$

  and analogously for sectors $\mathcal{W}_{ijjk}$ and $\mathcal{W}_{ijkl}$.

- $\mathbf{S}_{ij}\,RR$, $\mathbf{C}_{ikj}\,RR$, and $\mathbf{SC}_{ijk}\,RR$ are universal kernels [Catani, Grazzini, 9810389, 9908523], [Campbell, Glover, 9710255], [Berends, Giele, 1989].

- All limits commute.

NNLO-counterterm simplifications

- Further simplifications possible, thanks to idempotency relations

$$(1 - \mathbf{S}_i)\ \mathbf{SC}_{icd}\, RR\, \mathcal{W}_{ibcd} = 0\,, \qquad\qquad (1 - \mathbf{C}_{ij})\ \mathbf{CS}_{ijk}\, RR\, \mathcal{W}_{ijkd} = 0\,.$$

- Limits **SC** and **CS** disappear from $K^{(\mathbf{2})} + K^{(\mathbf{12})}$ (see also [Caola, Melnikov, Roentsch] about redundancy of **SC** in another scheme):

$$K^{(\mathbf{2})}_{ijkj} + K^{(\mathbf{12})}_{ijkj} \quad = \quad (1 - \mathbf{S}_i)(1 - \mathbf{C}_{ij})\Big[\mathbf{S}_{ik} + \mathbf{C}_{ijk}(1 - \mathbf{S}_{ik})\Big]\, RR\, \mathcal{W}_{ijkj}\,,$$

  and analogously for sectors $\mathcal{W}_{ijjk}$ and $\mathcal{W}_{ijkl}$.

- Still, since integrated $I^{(\mathbf{12})}$ and $I^{(\mathbf{2})}$ enter separately, they receive contributions from **SC** and **CS** (which however cancel in the sum).

Single-unresolved counterterm $\overline{K}^{(1)}$

- Candidate counterterm:

$$K^{(\mathbf{1})} = \sum_{abcd} K^{(\mathbf{1})}_{abcd} = \sum_{i,\, j\neq i} \left[\mathbf{S}_i + \mathbf{C}_{ij}\,(1-\mathbf{S}_i)\right] RR \sum_{k\neq i,j} \Big(\mathcal{W}_{ijjk} + \mathcal{W}_{ijkj} + \sum_{l\neq i,j,k} \mathcal{W}_{ijkl}\Big)$$

- Use factorisation properties of $\mathcal{W}_{abcd}$, and sum rules of $\mathcal{W}^{(\alpha\beta)}_{ab}$. For example, collinear:

$$\begin{aligned} K^{(\mathbf{1},\,\mathrm{c})} &= \sum_{i,\, j\neq i} (\mathbf{C}_{ij}\, RR)\big(\mathbf{C}_{ij}\, \mathcal{W}^{(\alpha\beta)}_{ij}\big) \sum_{k\neq i,j} \Big(\mathcal{W}_{[ij]k} + \mathcal{W}_{k[ij]} + \sum_{l\neq i,j,k} \mathcal{W}_{kl}\Big) \\ &= \sum_{i,\, j> i} (\mathbf{C}_{ij}\, RR) \sum_{k\neq i,j} \Big(\mathcal{W}_{[ij]k} + \mathcal{W}_{k[ij]} + \sum_{l\neq i,j,k} \mathcal{W}_{kl}\Big) \end{aligned}$$

- Map momenta $\{k\}_{1,\ldots,n+2}$ to momenta $\{\bar{k}\}_{1,\ldots,n+1}$. For example, collinear:

$$\overline{K}^{(\mathbf{1},\,\mathrm{c})} = \sum_{i,\, j> i} (\overline{\mathbf{C}}_{ij}\, RR) \sum_{k\neq i,j} \Big(\overline{\mathcal{W}}_{[ij]k} + \overline{\mathcal{W}}_{k[ij]} + \sum_{l\neq i,j,k} \overline{\mathcal{W}}_{kl}\Big) = \sum_{i,\, j> i} \sum_{k,\, l} (\overline{\mathbf{C}}_{ij}\, RR)\, \overline{\mathcal{W}}_{kl}\,.$$

- Analogously for soft. Full result:

$$\overline{K}^{(\mathbf{1})} = \sum_{k,l} \overline{\mathcal{W}}_{kl} \Bigg[\sum_{i,\, j> i} \overline{\mathbf{C}}_{ij}\, \big(1 - \overline{\mathbf{S}}_i - \overline{\mathbf{S}}_j\big)\, RR + \sum_i \overline{\mathbf{S}}_i\, RR\Bigg] = \sum_{k,l} \overline{K}^{(\mathbf{1})}_{kl}\,.$$

in each NLO sector

full structure of single-unres. singularities

Integrated single-unresolved counterterm $I^{(\mathbf{1})}$

$$I^{(\mathbf{1})}_{kl} \quad = \quad \overline{\mathcal{W}}_{kl}\,\frac{\varsigma_{n+2}}{\varsigma_{n+1}}\int d\Phi_{\mathrm{rad},1}\left[\sum_{i,\,j>i}\overline{\mathbf{C}}_{ij}\left(1-\overline{\mathbf{S}}_i-\overline{\mathbf{S}}_j\right)RR+\sum_i\overline{\mathbf{S}}_i\,RR\right].$$

- Same integral as the NLO integrated counterterm I (known to all orders in ϵ):

$$\begin{aligned} I^{(\mathbf{1})}_{kl}(\{\bar{k}\}) = & \\ & -\mathcal{N}_1\sum_p\frac{(4\pi)^{\epsilon-2}}{\bar{s}_{pr}^{\,\epsilon}}\,\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon\,\Gamma(2-3\epsilon)}\,\mathbb{C}\,R\big(\{\bar{k}\}\big)\,\overline{\mathcal{W}}_{kl}(\{\bar{k}\}) \\ & -\mathcal{N}_1\sum_{l,\,m\neq l}\frac{(4\pi)^{\epsilon-2}}{\bar{s}_{lm}^{\,\epsilon}}\,\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2\,\Gamma(2-3\epsilon)}\,R_{lm}(\{\bar{k}\})\,\overline{\mathcal{W}}_{kl}(\{\bar{k}\})\,, \end{aligned}$$

- Result valid in general (for FSR massless).
- Same $1/\epsilon$ structure as $RV\left(\{\bar{k}\}\right)\overline{\mathcal{W}}_{kl}\left(\{\bar{k}\}\right)$ sector by sector in the NLO phase space.
- $RV\,\overline{\mathcal{W}}_{kl}+I^{(\mathbf{1})}_{kl}$ finite in $d=4$ (analogously to NLO subtraction, virtual plus integrated counterterm).

Integration of the mixed double-unresolved counterterm $\overline{K}^{(\mathbf{12})}$

- Use factorisation properties of $\mathcal{W}_{abcd}$, and sum rules of $\mathcal{W}_{ab}^{(\alpha\beta)}$, as for $\overline{K}^{(\mathbf{1})}$.
- $I_{kl}^{(\mathbf{12})}$ in the end reduces to the collection of phase-space singularities of $I_{kl}^{(\mathbf{1})}$:

$$I_{kl}^{(\mathbf{12})} = \\ -\mathcal{N}_1 \sum_p \frac{(4\pi)^{\epsilon-2}}{\bar{s}_{pr}^{\epsilon}} \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon\,\Gamma(2-3\epsilon)} \mathbb{C}\Big[\overline{\mathbf{S}}_k + \overline{\mathbf{C}}_{kl}\left(1-\overline{\mathbf{S}}_k\right)\Big] R(\{\bar{k}\})\, \overline{\mathcal{W}}_{kl}(\{\bar{k}\}) \\ -\mathcal{N}_1 \sum_{l,\, m\neq l} \frac{(4\pi)^{\epsilon-2}}{\bar{s}_{lm}^{\epsilon}} \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2\,\Gamma(2-3\epsilon)} \Big[\overline{\mathbf{S}}_k + \overline{\mathbf{C}}_{kl}\left(1-\overline{\mathbf{S}}_k\right)\Big] R_{lm}(\{\bar{k}\})\, \overline{\mathcal{W}}_{kl}(\{\bar{k}\}) \,.$$

- Result valid in general (for FSR massless).
- Same $1/\epsilon$ structure as $\overline{K}_{kl}^{(\mathbf{RV})}(\{\bar{k}\})$ sector by sector in the NLO phase space.
- $\overline{K}_{kl}^{(\mathbf{RV})} - I_{kl}^{(\mathbf{12})}$ finite in $d=4$.

Pure double-unresolved counterterm $\overline{K}^{\,(\mathbf{2})}$

- Using sum rules, $\mathcal{W}$'s disappear from $\overline{K}^{\,(\mathbf{2})}$ and from its integral $I^{\,(\mathbf{2})}$. In the end:

$$\overline{K}^{\,(\mathbf{2})} = \sum_i \Bigg\{ \sum_{j>i} \overline{\mathbf{S}}_{ij} + \sum_{j>i}\sum_{k>j} \overline{\mathbf{C}}_{ijk} \left(1 - \overline{\mathbf{S}}_{ij} - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{jk}\right)$$
$$+ \sum_{j>i} \sum_{\substack{k>i \\ k\neq j}} \sum_{\substack{l>k \\ l\neq j}} \overline{\mathbf{C}}_{ijkl} \left(1 - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{jk} - \overline{\mathbf{S}}_{il} - \overline{\mathbf{S}}_{jl}\right)$$
$$+ \sum_{j\neq i} \sum_{\substack{k\neq i \\ k>j}} \overline{\mathbf{SC}}_{ijk} \left(1 - \overline{\mathbf{S}}_{ij} - \overline{\mathbf{S}}_{ik}\right) \Bigg(1 - \overline{\mathbf{C}}_{ijk} - \sum_{l\neq i,j,k} \overline{\mathbf{C}}_{iljk}\Bigg)$$
$$+ \sum_{j>i} \sum_{k\neq i,j} \overline{\mathbf{CS}}_{ijk} \left(1 - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{jk}\right) \Bigg(1 - \overline{\mathbf{C}}_{ijk} - \sum_{l\neq i,j,k} \overline{\mathbf{C}}_{ijkl}\Bigg) \Bigg\} RR\,,$$

- Analytic integration of a set of universal NNLO kernels with no $\mathcal{W}$ functions.

- As at NLO, different kernels and/or different terms in the same kernel can be mapped/parametrised differently to ease integration.

Mappings from NNLO to Born kinematics

- Different kernels and/or different terms in the same kernel can be mapped/parametrised differently to ease integration.

- One of the choices $\{k\} \to \{\bar{k}\}^{(abcd)}$:

$$\bar{k}_n^{(abcd)} = k_n \,, \qquad n \neq a, b, c, d \,,$$

$$\bar{k}_c^{(abcd)} = k_a + k_b + k_c - \frac{s_{abc}}{s_{ad} + s_{bd} + s_{cd}} \, k_d \,, \qquad \bar{k}_d^{(abcd)} = \frac{s_{abcd}}{s_{ad} + s_{bd} + s_{cd}} \, k_d \,,$$

with $s_{abcd} = s_{ab} + s_{ac} + s_{ad} + s_{bc} + s_{bd} + s_{cd}$ and $\bar{k}_c^{(abcd)} + \bar{k}_d^{(abcd)} = k_a + k_b + k_c + k_d$.

- This is used to define double-collinear $\overline{\mathbf{C}}_{ijk} \, RR$ and (part of) the double-soft $\overline{\mathbf{S}}_{ij} \, RR$ counterterms:

$$\overline{\mathbf{S}}_{ij} \, RR = \frac{\mathcal{N}_1^2}{2} \sum_{\substack{c \neq i,j \\ d \neq i,j}} \mathcal{I}_{cd}^{(ij)} B_{cd}\Big(\{\bar{k}\}^{(ijcd)}\Big) + \ldots,$$

$$\overline{\mathbf{C}}_{ijk} \, RR = \frac{\mathcal{N}_1^2}{s_{ijk}^2} \Big[P_{ijk} \, B\Big(\{\bar{k}\}^{(ijkr)}\Big) + Q_{ijk}^{\mu\nu} \, B_{\mu\nu}\Big(\{\bar{k}\}^{(ijkr)}\Big) \Big]$$

(argument of the collinear kernels are $z_a = s_{ar}/(s_{ir} + s_{jr} + s_{kr})$, $a = i, j, k$)

Integration of the pure double-unresolved counterterm $\overline{K}^{(\mathbf{2})}$ (I)

- Catani-Seymour variables $y, z, y', z', x' \in [0,1]$ for mapping $\{k\} \to \{\bar{k}\}^{(abcd)}$:

$$s_{ab} = y'\, y\, s_{abcd}\,, \qquad s_{cd} = \big(1-y'\big)\,(1-y)\,(1-z)\;s_{abcd}\,,$$
$$s_{ac} = z'\,\big(1-y'\big)\;y\,s_{abcd}\,, \qquad s_{bc} = \big(1-y'\big)\,\big(1-z'\big)\;y\,s_{abcd}\,,$$
$$s_{ad} = (1-y)\left[y'\,\big(1-z'\big)\,(1-z) + z'z - 2\,\big(1-2x'\big)\sqrt{y'z'\,(1-z')\;z\,(1-z)}\right] s_{abcd}\,,$$
$$s_{bd} = (1-y)\left[y'z'\,(1-z) + \big(1-z'\big)\,z + 2\,\big(1-2x'\big)\sqrt{y'z'\,(1-z')\;z\,(1-z)}\right] s_{abcd}\,,$$

- Phase-space factorisation:

$$d\Phi_{n+2} \quad = \quad d\Phi_{n}^{(abcd)}\, d\Phi_{\mathrm{rad},2}^{(abcd)}\,,$$

$$\begin{aligned}
\int d\Phi_{\mathrm{rad},2}^{(abcd)} \quad &= \quad \int d\Phi_{\mathrm{rad},2}\left(s_{abcd};\, y, z, \phi, y', z', x'\right) \\
&= \quad N^2(\epsilon)\,(s_{abcd})^{2-2\epsilon}\int_0^1 dx' \int_0^1 dy' \int_0^1 dz' \int_0^\pi d\phi\,\left(\sin\phi\right)^{-2\epsilon} \int_0^1 dy \int_0^1 dz \\
&\quad \times \left[4\,x'\,\big(1-x'\big)\;y'\,\big(1-y'\big)^2\;z'\,\big(1-z'\big)\;y^2(1-y)^2\,z\,(1-z)\right]^{-\epsilon} \\
&\quad \times \left[x'\,\big(1-x'\big)\right]^{-1/2}\big(1-y'\big)\;y\,(1-y)\,.
\end{aligned}$$

Integration of the double-unresolved counterterm $\overline{K}^{(\mathbf{2})}$ (II)

- Example double-soft $q\bar{q}$: each term of the eikonal sum parametrised and mapped differently with $(abcd) = (ijlm)$.

$$\int d\Phi_{\mathrm{rad},2}\, \overline{\mathbf{S}}_{ij}\, RR = \mathcal{N}_1^2\, T_R \sum_{l,m=1}^{2} B_{lm}\Big(\{\bar{k}\}^{(ijlm)}\Big) \int d\Phi_{\mathrm{rad},2}^{(ijlm)}\, \frac{s_{il}s_{jm} + s_{im}s_{jl} - s_{ij}s_{lm}}{s_{ij}^2\,(s_{il}+s_{jl})\,(s_{im}+s_{jm})}$$

$$= \mathcal{N}_1^2\, B\, T_R\, C_F\, \frac{8}{s^2} \int d\Phi_{\mathrm{rad},2}\,(s; y, z, \phi, y', z', x')\, \frac{z'\,(1-z')}{y^2 y'^2}\, \frac{y'\,(1-z)}{y'\,(1-z)+z}$$

$$= B\left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F \left(\frac{\mu^2}{s}\right)^{2\epsilon} \left[-\frac{1}{3\epsilon^3} - \frac{17}{9\epsilon^2} + \frac{1}{\epsilon}\left(\frac{7}{18}\pi^2 - \frac{232}{27}\right) + \left(\frac{38}{9}\zeta_3 + \frac{131}{54}\pi^2 - \frac{2948}{81}\right)\right] + \mathcal{O}(\epsilon).$$

- Double-collinear $q \to qq'\bar{q}'$, parametrised and mapped with $(abcd) = (ijkr)$.

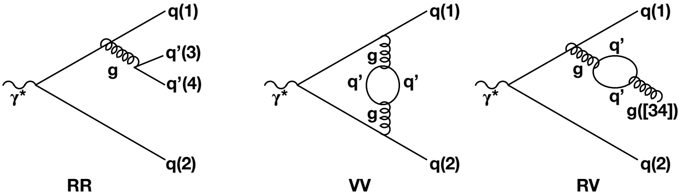
$$\int d\Phi_{\mathrm{rad},2}^{(ijkr)}\, \overline{\mathbf{C}}_{ijk} RR \;=\; \mathcal{N}_1^2 T_R C_F B \int d\Phi_{\mathrm{rad},2}^{(ijkr)}\, \frac{1}{2 s_{ijk} s_{ik}} \left[-\frac{t_{ik,j}^2}{s_{ik}s_{ikj}} + \frac{4z_j + (z_i - z_k)^2}{z_i + z_k} + (1-2\epsilon)\left(z_i + z_k - \frac{s_{ik}}{s_{ikj}}\right)\right]$$

$$= B\left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F \left(\frac{\mu^2}{s}\right)^{2\epsilon} \left[-\frac{1}{3\epsilon^3} - \frac{31}{18\epsilon^2} + \frac{1}{\epsilon}\left(\frac{1}{2}\pi^2 - \frac{889}{108}\right) + \left(\frac{80}{9}\zeta_3 + \frac{31}{12}\pi^2 - \frac{23941}{648}\right)\right] + \mathcal{O}(\epsilon)\,.$$

- Other kernels more complicated, but manageable analytically (ongoing).

A proof-of-concept example

- $T_R C_F$ NNLO contribution to the total cross section for $e^+e^- \to q\bar{q}$ (analytic matrix elements from [Hamberg, van Neerven, Matsuura, 1991], [Gehrmann De Ridder, Gehrmann, Glover, 0403057], [Ellis, Ross, Terrano, 1980])



$$
\begin{aligned}
VV \;&=\; B\left(\frac{\alpha_S}{2\pi}\right)^2 T_R\, C_F \Bigg\{ \left(\frac{\mu^2}{s}\right)^{2\epsilon} \left[\frac{1}{3\epsilon^3} + \frac{14}{9\epsilon^2} + \frac{1}{\epsilon}\left(-\frac{11}{18}\pi^2 + \frac{353}{54}\right) + \left(-\frac{26}{9}\zeta_3 - \frac{77}{27}\pi^2 + \frac{7541}{324}\right)\right] \\
&\qquad + \left(\frac{\mu^2}{s}\right)^{\epsilon} \left[-\frac{4}{3\epsilon^3} - \frac{2}{\epsilon^2} + \frac{1}{\epsilon}\left(\frac{7}{9}\pi^2 - \frac{16}{3}\right) + \left(\frac{28}{9}\zeta_3 + \frac{7}{6}\pi^2 - \frac{32}{3}\right)\right] \Bigg\}, \\
\int d\Phi_{\rm rad}\, RV \;&=\; \frac{\alpha_S}{2\pi}\,\frac{1}{\epsilon}\,\frac{2}{3}\, T_R \int d\Phi_{\rm rad}\, R \\
&=\; B\left(\frac{\alpha_S}{2\pi}\right)^2 T_R\, C_F \left(\frac{\mu^2}{s}\right)^{\epsilon} \left[\frac{4}{3\epsilon^3} + \frac{2}{\epsilon^2} + \frac{1}{\epsilon}\left(-\frac{7}{9}\pi^2 + \frac{19}{3}\right) + \left(-\frac{100}{9}\zeta_3 - \frac{7}{6}\pi^2 + \frac{109}{6}\right)\right], \\
\int d\Phi_{\rm rad,2}\, RR \;&=\; B\left(\frac{\alpha_S}{2\pi}\right)^2 T_R\, C_F \left(\frac{\mu^2}{s}\right)^{2\epsilon} \left[-\frac{1}{3\epsilon^3} - \frac{14}{9\epsilon^2} + \frac{1}{\epsilon}\left(\frac{11}{18}\pi^2 - \frac{407}{54}\right) + \left(\frac{134}{9}\zeta_3 + \frac{77}{27}\pi^2 - \frac{11753}{324}\right)\right].
\end{aligned}
$$

Integrated counterterms: $I^{(\mathbf{2})}$, $I^{(\mathbf{1})}$, $I^{(\mathbf{12})}$, $I^{(\mathbf{RV})}$

- $I^{(\mathbf{2})}$: in the case at hand only $\overline{\mathbf{S}}_{34}\,RR$, $\overline{\mathbf{C}}_{134}\,RR$, $\overline{\mathbf{C}}_{234}\,RR$ are non-zero, so

$$I^{(\mathbf{2})} = \int d\Phi_{\rm rad,2}\left[\overline{\mathbf{S}}_{34} + \overline{\mathbf{C}}_{134}\,(1-\overline{\mathbf{S}}_{34}) + \overline{\mathbf{C}}_{234}\,(1-\overline{\mathbf{S}}_{34})\right] RR$$
$$= B\left(\frac{\alpha_{\rm S}}{2\pi}\right)^2 T_R C_F \left(\frac{\mu^2}{s}\right)^{2\epsilon}\left[-\frac{1}{3\epsilon^3} - \frac{14}{9\epsilon^2} + \frac{1}{\epsilon}\left(\frac{11}{18}\pi^2 - \frac{425}{54}\right) + \left(\frac{122}{9}\zeta_3 + \frac{74}{27}\pi^2 - \frac{12149}{324}\right)\right].$$

- $I^{(\mathbf{1})}$ and $I^{(\mathbf{12})}$ from general formulae above:

$$I^{(\mathbf{1})}_{hq} = -\frac{\alpha_{\rm S}}{2\pi}\left(\frac{\mu^2}{s}\right)^{\epsilon}\frac{2}{3}\,T_R\left(\frac{1}{\epsilon} - \ln\bar\eta_{[34]r} + \frac{8}{3}\right) R\,\overline{\mathcal{W}}_{hq} + \mathcal{O}(\epsilon)\,,$$
$$I^{(\mathbf{12})}_{hq} = \frac{\alpha_{\rm S}}{2\pi}\left(\frac{\mu^2}{s}\right)^{\epsilon}\frac{2}{3}\,T_R\left(\frac{1}{\epsilon} - \ln\bar\eta_{[34]r} + \frac{8}{3}\right)\left[\bar{\mathbf{S}}_h + \overline{\mathbf{C}}_{hq}\,(1-\bar{\mathbf{S}}_h)\right] R\,\overline{\mathcal{W}}_{hq} + \mathcal{O}(\epsilon)\,.$$

- Finiteness of contributions in the $(n+1)$-body phase space, sector by sector:

$$RV\,\overline{\mathcal{W}}_{hq} + I^{(\mathbf{1})}_{hq} = -\frac{\alpha_{\rm S}}{2\pi}\,\frac{2}{3}\,T_R\left(\ln\frac{\mu^2}{s_{34r}} + \frac{8}{3}\right) R\,\overline{\mathcal{W}}_{hq} + \mathcal{O}(\epsilon)\,.$$
$$\overline{K}^{(\mathbf{RV})}_{hq} - I^{(\mathbf{12})}_{hq} = -\frac{\alpha_{\rm S}}{2\pi}\,\frac{2}{3}\,T_R\left(\ln\frac{\mu^2}{s_{34r}} + \frac{8}{3}\right)\left[\bar{\mathbf{S}}_h + \overline{\mathbf{C}}_{hq}\,(1-\bar{\mathbf{S}}_h)\right] R\,\overline{\mathcal{W}}_{hq} + \mathcal{O}(\epsilon)\,.$$

- Real-virtual counterterm:

$$I^{(\mathbf{RV})} = \frac{\alpha_{\rm S}}{2\pi}\,\frac{2}{3}\,\frac{1}{\epsilon}\,T_R\int d\Phi_{\rm rad}\left[\overline{\mathbf{S}}_{[34]} + \overline{\mathbf{C}}_{1[34]}\,(1-\overline{\mathbf{S}}_{[34]}) + \overline{\mathbf{C}}_{2[34]}\,(1-\overline{\mathbf{S}}_{[34]})\right] R$$
$$= B\left(\frac{\alpha_{\rm S}}{2\pi}\right)^2 T_R\,C_F\left(\frac{\mu^2}{s}\right)^{\epsilon}\left[\frac{4}{3\epsilon^3} + \frac{2}{\epsilon^2} - \frac{1}{\epsilon}\left(\frac{7}{9}\pi^2 - \frac{20}{3}\right) - \left(\frac{100}{9}\zeta_3 + \frac{7}{6}\pi^2 - 20\right)\right] + \mathcal{O}(\epsilon)\,,$$

Collection of results

- Subtracted double-virtual (analytic):

$$\begin{aligned} VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})} &= B\left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F \left(\frac{8}{3}\zeta_3 - \frac{1}{9}\pi^2 - \frac{44}{9} - \frac{4}{3}\ln\frac{\mu^2}{s}\right) \\ &= B\left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F \times 0.01949914. \qquad \longleftarrow \quad \mu/\sqrt{s} = 0.35 \end{aligned}$$

- Subtracted real-virtual and double-real (numerical in $d = 4$):

$$\begin{aligned} \int d\Phi_{\rm rad}\left(RV + I^{(\mathbf{1})} - (K^{(\mathbf{RV})} - I^{(\mathbf{12})})\right) &= B\left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F \times (-0.90635 \pm 0.00011), \\ \int d\Phi_{\rm rad,2}\left(RR - K^{(\mathbf{1})} - K^{(\mathbf{2})} - K^{(\mathbf{12})}\right) &= B\left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F \times (+2.29491 \pm 0.00038). \end{aligned}$$

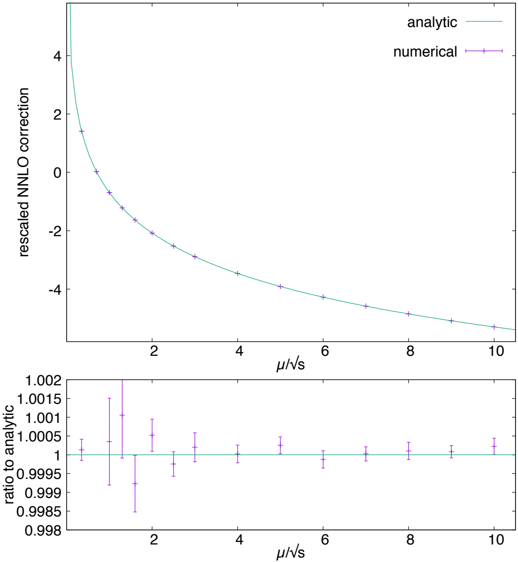
- Rescaled NNLO coefficient, from the subtraction method

$$\frac{1}{\left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F}\,\frac{\sigma_{\rm NNLO}}{\sigma_{\rm LO}} = 1.40806 \pm 0.00040.$$

- Analytic result

$$-\frac{11}{2} + 4\zeta_3 - \ln\frac{\mu^2}{s} = 1.40787186.$$

Renormalisation-scale dependence



NNLO summary

- Sector functions at NNLO can be engineered to factorise NLO-sector structure.
- Straightforward analytic integration of single-unresolved counterterm $\overline{K}^{(\mathbf{1})}$ (NLO complexity) and cancellation of $1/\epsilon$ poles of RV NLO-sector by NLO-sector.
- Straightforward analytic integration of mixed double-unresolved counterterm $\overline{K}^{(\mathbf{12})}$ (NLO complexity) and cancellation of $1/\epsilon$ poles of $\overline{K}^{(\mathbf{RV})}$ NLO-sector by NLO-sector.
- Sector-function sum rules to simplify as much as possible pure double-unresolved integrands $\overline{K}^{(\mathbf{2})}$: only sums of universal kernels.
- Exploit full freedom in mapping and parametrisation of each contribution separately.

Status

- Method for the moment applied to FSR only and massless.
- Analytic integration of $\overline{K}^{(\mathbf{2})}$ to be finished. Most probably possible without IBP methods for the massless case.
- Real-virtual counterterms to be integrated (simpler than $\overline{K}^{(\mathbf{2})}$).
- Ongoing implementation in a differential code.

Thank you

Backup

Soft/collinear commutation at NLO

- Soft limit $\mathbf{S}_i$ ($k_i^\mu \to 0$): $s_{ia}/s_{ib} \to$ constant, $s_{ia}/s_{bc} \to 0, \forall\, a, b, c \neq i$.
- Collinear limit $\mathbf{C}_{ij}$ ($k_\perp \to 0$): $s_{ij}/s_{ia} \to 0$, $s_{ij}/s_{jb} \to 0$, $s_{ij}/s_{ab} \to 0, \forall\, a, b \neq i, j$. $s_{ia}/s_{ja} \to$ independent of a.
- Commutation in case $i =$ gluon and $j =$ quark.
- Altarelli-Parisi collinear kernel involved is $P_{ij}(x_i) = [1 + (1 - x_i)^2]/x_i$, with $x_i = s_{ir}/(s_{ir} + s_{jr})$, with arbitrary $r \neq i, j$.

$$
\begin{aligned}
\mathbf{S}_i\, R &= -\mathcal{N}_1 \sum_{\substack{l \neq i \\ m \neq i}} \frac{s_{lm}}{s_{il} s_{im}} B_{lm} \\
\Longrightarrow \quad \mathbf{C}_{ij}\, \mathbf{S}_i\, R &= -2\,\mathcal{N}_1 \sum_{l \neq i,j} \frac{s_{jl}}{s_{il} s_{ij}} B_{lj} \quad = \quad -\,2\,\mathcal{N}_1 \frac{s_{jr}}{s_{ir} s_{ij}} (-C_{f_j} B), \\
\mathbf{C}_{ij}\, R &= \mathcal{N}_1 \frac{1}{s_{ij}} C_{f_j} B \frac{1 + [1 - s_{ir}/(s_{ir} + s_{jr})]^2}{s_{ir}/(s_{ir} + s_{jr})} \\
\Longrightarrow \quad \mathbf{S}_i \mathbf{C}_{ij} R &= -2\,\mathcal{N}_1 \frac{s_{jr}}{s_{ir} s_{ij}} (-C_{f_j} B).
\end{aligned}
$$

Soft counterterm in FKS

- The soft FKS counterterm does not feature gluon energy, thus it reduces to an angular integral:

$$I^{\mathrm{s}}_{\mathrm{FKS}} \propto \sum_{lm} \int d\cos\theta \, d\phi (\sin\phi \sin\theta)^{-2\epsilon} \frac{1-\cos\theta_{lm}}{(1-\cos\theta_{li})(1-\cos\theta_{mi})}.$$

- Doable (actually relevant to angular-ordering), but not maximally easy: relations among θ_{lm}, θ_{li} and θ_{mi} are non-trivial in terms of integration variables.

- Analogous features at NNLO may be much more severe.

Cancellation of virtual NLO poles

- Integrated counterterm I computed at all orders in ϵ.
- ϵ expansion:

$$
\begin{aligned}
I(\{\bar{k}\}) \;=\; & \frac{\alpha_S}{2\pi}\left(\frac{\mu^2}{s}\right)^{\epsilon}\Bigg\{\Bigg[B(\{\bar{k}\})\sum_k\left(\frac{C_{f_k}}{\epsilon^2}+\frac{\gamma_k}{\epsilon}\right)+\sum_{k,\,l\neq k}B_{kl}(\{\bar{k}\})\,\frac{1}{\epsilon}\ln\bar{\eta}_{kl}\Bigg] \\
& +\Bigg[B(\{\bar{k}\})\sum_k\Bigg(\delta_{f_k g}\,\frac{C_A+4\,T_R\,N_f}{6}\left(\ln\bar{\eta}_{kr}-\frac{8}{3}\right) \\
& +\delta_{f_k g}\,C_A\left(6-\frac{7}{2}\zeta_2\right)+\delta_{f_k\{q,\bar{q}\}}\,\frac{C_F}{2}\left(10-7\zeta_2+\ln\bar{\eta}_{kr}\right)\Bigg) \\
& +\sum_{k,\,l\neq k}B_{kl}\left(\{\bar{k}\}\right)\ln\bar{\eta}_{kl}\left(2-\frac{1}{2}\ln\bar{\eta}_{kl}\right)\Bigg]\Bigg\}.
\end{aligned}
$$

- $\bar{\eta}_{ab}=\bar{s}_{ab}/s$, and $\gamma_k \;=\; \delta_{f_k g}\,\frac{11C_A-4\,T_R N_f}{6}+\delta_{f_k\{q,\bar{q}\}}\,\frac{3}{2}C_F$.
- Same structure of ϵ singularities as V (up to a sign).

NNLO sector-function sum rules for composite limits

$$\mathbf{S}_i\,\mathbf{C}_{ijk}\,\left(\mathcal{W}_{ij}^{(\alpha\beta)}+\mathcal{W}_{ik}^{(\alpha\beta)}\right)=1\,,$$

$$\mathbf{S}_{ij}\,\mathbf{C}_{ijk}\sum_{ab\,\in\,\mathrm{perm}(ij)}\left(\mathcal{W}_{abbk}+\mathcal{W}_{akbk}\right)=1\,,\qquad \mathbf{S}_{ik}\,\mathbf{C}_{ijkl}\,\left(\mathcal{W}_{ijkl}+\mathcal{W}_{klij}\right)=1\,.$$

$$\mathbf{SC}_{ijk}\,\mathbf{S}_{ij}\sum_{b\neq i}\mathcal{W}_{ibjk}=1\,,\qquad \mathbf{CS}_{ijk}\,\mathbf{S}_{ik}\sum_{d\neq i,k}\mathcal{W}_{ijkd}=1\,,$$

$$\mathbf{CS}_{ijk}\,\mathbf{C}_{ijk}\,\left(\mathcal{W}_{ijkj}+\mathcal{W}_{jiki}\right)=1\,,\qquad \mathbf{CS}_{ijk}\,\mathbf{C}_{ijkl}\,\left(\mathcal{W}_{ijkl}+\mathcal{W}_{jikl}\right)=1\,,$$

$$\mathbf{CS}_{ijk}\,\mathbf{C}_{ijk}\,\mathbf{S}_{ik}\,\mathcal{W}_{ijkj}=1\,,\qquad \mathbf{CS}_{ijk}\,\mathbf{C}_{ijkl}\,\mathbf{S}_{ik}\,\mathcal{W}_{ijkl}=1\,,$$

$$\mathbf{SC}_{ijk}\,\mathbf{C}_{ijk}\sum_{ab\,\in\,\mathrm{perm}(jk)}\left(\mathcal{W}_{iaab}+\mathcal{W}_{iaba}\right)=1\,,\qquad \mathbf{SC}_{ikl}\,\mathbf{C}_{ijkl}\,\left(\mathcal{W}_{ijkl}+\mathcal{W}_{ijlk}\right)=1\,,$$

$$\mathbf{SC}_{ijk}\,\mathbf{C}_{ijk}\,\mathbf{S}_{ik}\,\left(\mathcal{W}_{ijkj}+\mathcal{W}_{ikkj}\right)=1\,,\qquad \mathbf{SC}_{ijk}\,\mathbf{C}_{ijkl}\,\mathbf{S}_{ik}\,\mathcal{W}_{ijkl}=1\,.$$