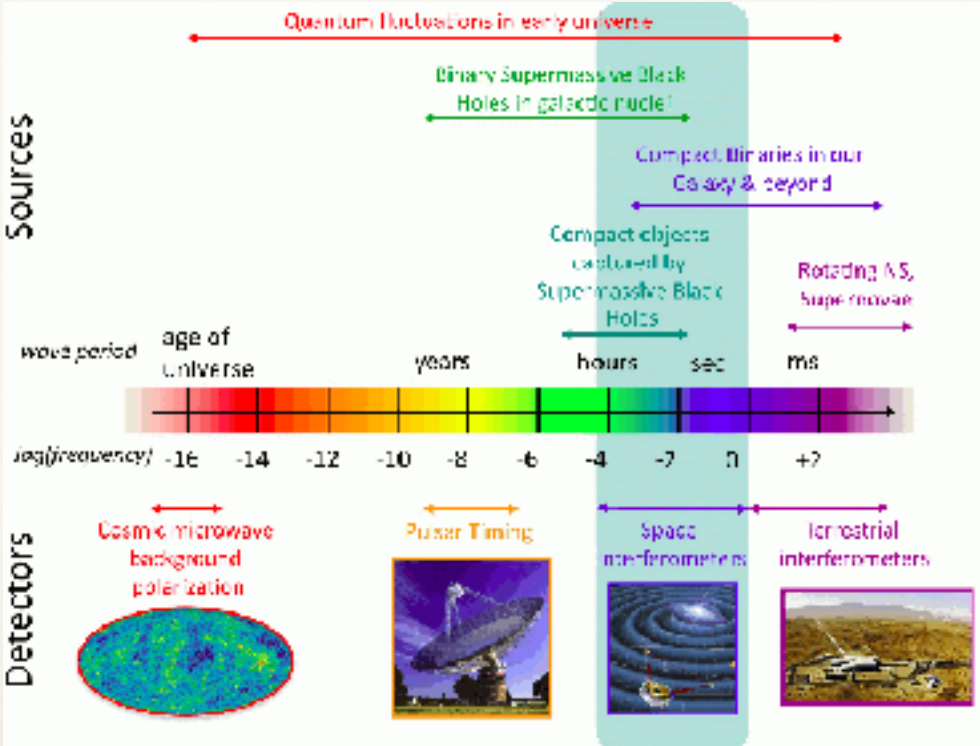

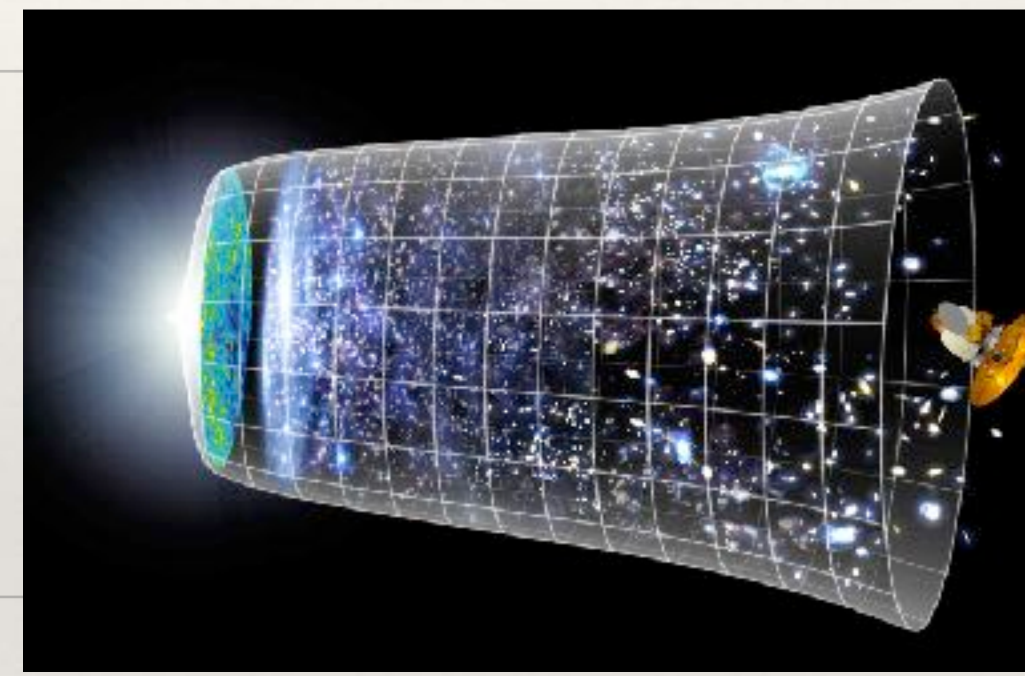

Detecting stochastic GW signal

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Basic knowledge

Let us introduce some notations:

$$d(t) = n(t) + s(t, \vec{\lambda})$$

Observed data

Noise

Signal (parametrised)

We assume that the noise and the signal are independent.

In order to detect the signal we have to know the noise properties. The usual assumptions are that the noise is stationary and Gaussian:

- a large number of small disturbances combined with counting noise in the large-number limit, the central limit theorem suggests that the noise distribution can be approximated by a multi-variate normal (Gaussian) distribution

$$p(n) = \frac{1}{det(2\pi C_n)} e^{-\frac{1}{2}\sum_{i,j} n_i (C_n^{-1})_{ij} n_j}$$

$$(C_n)_{ij} = < n_i n_j > - < n_i >< n_j >, \quad n_i = n(t_i)$$

noise variance-covariance matrix

What do we measure

The signal could be deterministic or stochastic.
The GW strain is described by a tensor h_{ij}, which is convolved with the instrument response function (different for each project: LIGO/VIRGO, LISA, PTA

$$h_{ab} = h_+ e^+_{ab} + h_\times e^\times_{ab}$$

Polarisation tensor

$$h_{ab} = \int df \int d^2\Omega_{\hat{n}} h_{ab}(f,\hat{n}) e^{2\pi f(t+\hat{n}\vec{x}/c)}$$

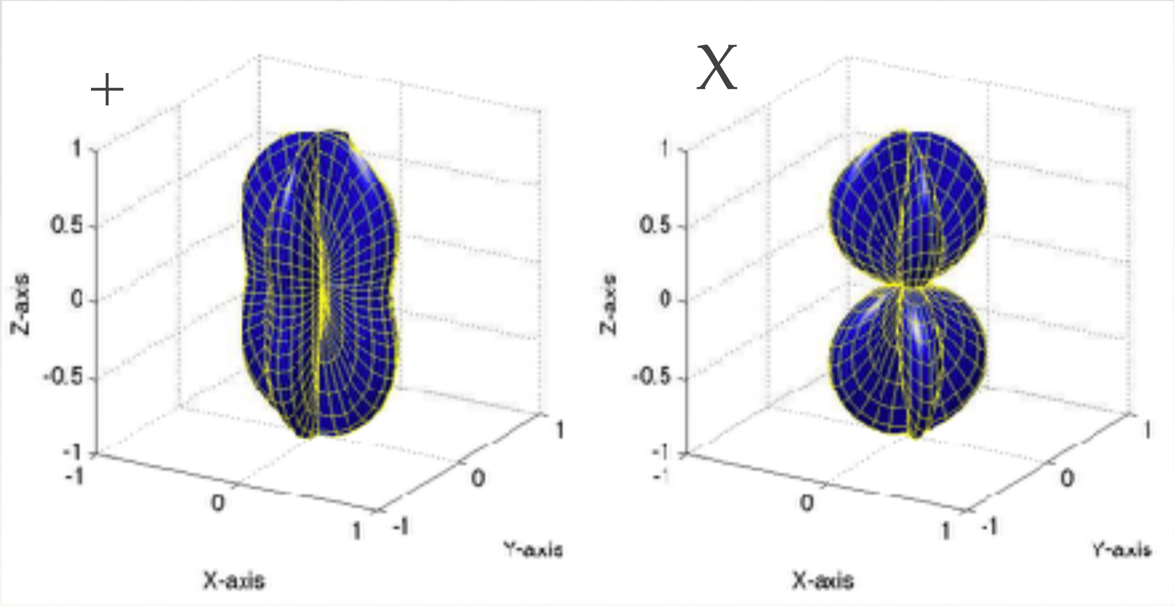
$$h(t) = \mathbf{R} * \mathbf{h} = \int df \int d^2\Omega_{\hat{n}} R^{ab}(t,f,\hat{n}) h_{ab}(f,\hat{n}) e^{2\pi f t}$$

Response function: depends on the method of GW measurement

$$R_{+,\times} = R^{ab}(t,f,\hat{n}) e^{+,\times}_{ab}$$

$$\mathcal{R} = \sqrt{R_+^2 + R_\times^2}$$

LIGO



Analysis method

If the signal is present in the data

$$n(t) = d(t) - s(t, \vec{\lambda})$$

The likelihood of observing data (d) should be consistent with a draw from the noise distribution p_n

Likelihood: $$p(d(t)|\bar{s}(t, \lambda) = p(d(t) - s(t, \vec{\lambda})) = p_n$$

- Frequentist analysis:
  - The detection is based on the maximising of the detection statistic. The detection threshold is set by a desired false alarm probability.
  - The signal's parameters are deterministic but corrupted by noise we need to estimate the parameters (estimator) based on the maximisation of a detection statistic (e.g. maximum likelihood). Estimator could take un-physical values and its distribution obtained from the multiple outcome of the experiment (different noise realisations).
  - We construct confidence interval from the pdf of an estimator and compute p-value for hypothesis testing.

Analysis method

- Bayesian approach
  - The signal's parameters are random variables. We assign probabilities for the parameters and to a given hypothesis
  - We need to specify a prior degree of belief in a particular hypothesis and parameters.
  - We use measurements and Bayes theorem to update the prior probability -> likelihood & prior -> posterior probability
  - Parameters are described by posterior distribution functions
  - The hypothesis testing is done through posterior odd ratio
  - Detection is somewhat subjective (based on the value for the odd ratio)

Likelihood

Prior

$$\underbrace{p(\lambda_a|d, M_a)}_{\text{Posterior}} = \frac{\overbrace{p(d|\lambda_a, M_a)}^{\text{Likelihood}}\overbrace{p(\lambda_a|M_a)}^{\text{Prior}}}{\underbrace{p(d|M_a)}_{\text{Evidence}}}$$

Posterior

Evidence

$$p(d|M_a) = \int p(d|\lambda_a, M_a)p(\lambda_a|M_a)\, d\lambda_a$$

Model selection

Odd ratio - used to check which model is supported by the data

$$O_{a,b} = \frac{p(M_a|d)}{p(M_b|d)} = \frac{p(M_a)}{p(M_b)}\frac{p(d|M_a)}{p(d|M_b)} \qquad B_{a,b} = \frac{p(d|M_a)}{p(d|M_b)}$$

Bayes factor

Choosing prior: Non trivial, wrong prior could give the erroneous outcome. The prior includes the range of parameters and their distribution (if known, if not - use non - informative prior).

Computation of evidence: Laplace approximation for evidence

$$p(d|M_a) = \int p(d|\lambda_a, M_a)p(\lambda_a|M_a)\, d\lambda_a \approx p(d|\lambda_a^{ML}, M_a)\frac{\Delta V_{M_a}}{V_{M_a}}$$

$$2\ln B_{1,0} \approx 2\ln(\Lambda_{ML}(d)) + 2\ln\left(\frac{\Delta V_1/V_1}{\Delta V_0/V_0}\right)$$

Basics behind detecting stochastic GWs

The key property that allows one to distinguish a a stochastic gravitational-wave background from instrumental noise is that the gravitational-wave signal is correlated across multiple detectors while instrumental noise typically is not.

$$d_1 = s + n_1, \quad d_2 = s + n_2$$ Output from two detectors

Assume two co-located detectors

Assuming noise has zero-mean

Compute correlation between output of two detectors

$$< C_{12} >=< d_1 d_2 >=< s^2 > + < n_1 n_2 >$$

$$< n_1 n_2 >= 0, \;\rightarrow\; < C_{12} >=< s^2 >= S_h \longrightarrow$$ Power of the stochastic GW signal

Likelihood

Likelihood ("h"- model of the signal "s"

$$p(d|S_{n_1}, S_{n_2}, h, M_1) = \frac{1}{\sqrt{2\pi C}} e^{-\frac{1}{2}\sum_{I_i J_j}(d_{I_i} - h_{I_i})(C^{-1})_{I_i J_j}(d_{J_j} - h_{J_j})}$$

$$C = \begin{bmatrix} < C_{11} > & \boxed{< C_{12} >} \\ \boxed{< C_{21} >} & < C_{22} > \end{bmatrix}$$

Zero if noise is uncorrelated

For the stochastic signal we are not interested in the amplitude of the SGW at each sample but in the power of the signal. We assume that amplitude is distributed as a Gaussian random deviate:

$$p(h|S_h, M_1) = \frac{1}{2\pi S_h} e^{-\frac{h^2}{2S_h}}$$

We can marginalise the likelihood over amplitude h:

$$p(d|S_{n_1}, S_{n_2}, S_h, M_1) = \frac{1}{\sqrt{2\pi C}} e^{-\frac{1}{2}\sum_{I_i J_j}(d_{I_i})^T (C^{-1})_{I_i J_j}(d_{J_j})}$$

$$C = \begin{bmatrix} (S_n + S_h)_{I_i, I_j} & (S_h)_{I_i, J_j} \\ (S_h)_{J_i, I_j} & (S_n + S_h)_{J_i, J_j} \end{bmatrix}$$

Moving to frequency domain (non-white noise)

In case of the coloured noise we can work in the frequency domain

Use power spectral density

$$< \tilde{n}_I(f)\tilde{n}_I^*(f') >= \frac{1}{2}\delta(f-f')S_{n_I}(f)$$

$$< \tilde{h}_I(f)\tilde{h}_I^*(f') >= \frac{1}{2}\delta(f-f')S_{h_I}(f)$$

$$Var(h) = \int_0^\infty df S_h(f)$$

And the likelihood

$$p(d|S_{n_1}, S_{n_2}, S_h, M_1) = \Pi_{k=0}^{N/2-1} \frac{1}{\sqrt{2\pi C(f_k)}} e^{-\frac{1}{2}\sum_{I,J} \tilde{d}^*(f_k)\left(C^{-1}(f_k)\right)_{I,J}\tilde{d}_J(f_k)}$$

$$C(f_k) = \frac{T}{4}\begin{bmatrix} S_{n_1}(f_k) + S_h(f_k) & S_h(f_k) \\ S_h(f_k) & S_{n_2}(f_k) + S_h(f_k) \end{bmatrix}$$

Loosing simplifications

Next we need to discuss prior and response function

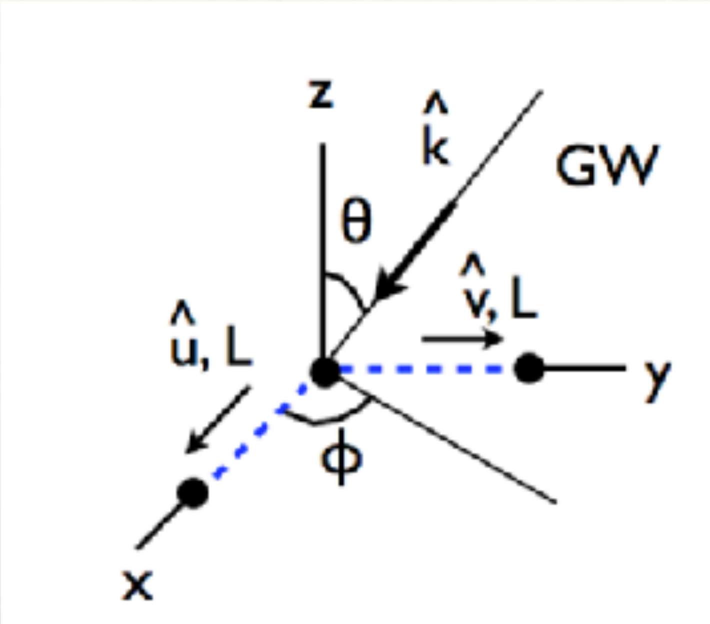
Prior for the noise $p(S_n) \propto 1/S_n$

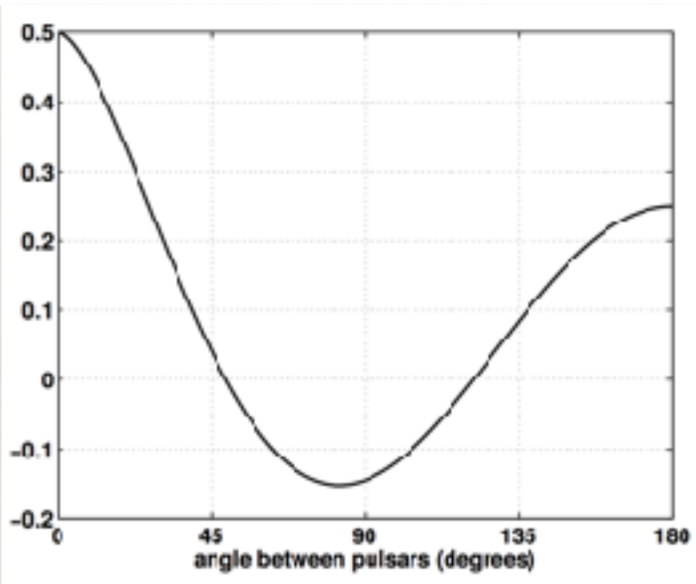
Prior for the signal $p(S_h) \propto S_h$

For not-colocated detectors: $h_I \neq h_J$

$$h_I(t) = \mathbf{R_I} * \mathbf{h_I} = \int df \int d^2\Omega_{\hat{n}} R_I^{ab}(t, f, \hat{n}) h_{ab}(f, \hat{n}) e^{2\pi f t}$$

Overlap reduction function





For LIGO/VIRGO $fL/c \ll 1$ $R_{ab} = \frac{1}{2}(u^a u^b - v^a v^b)$

Isotropic background

$$< h_I(t) h_J(t') > = \frac{1}{2} \int df \, \Gamma_{IJ}(f) S_h(f) e^{i2\pi f(t-t')}$$

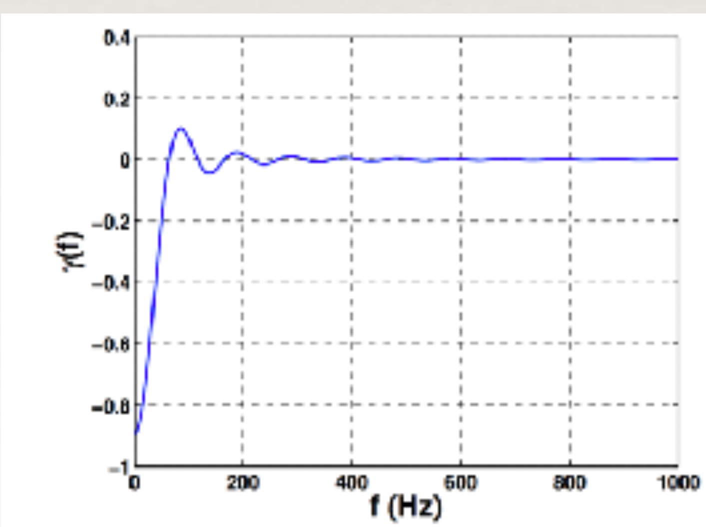
Overlap function

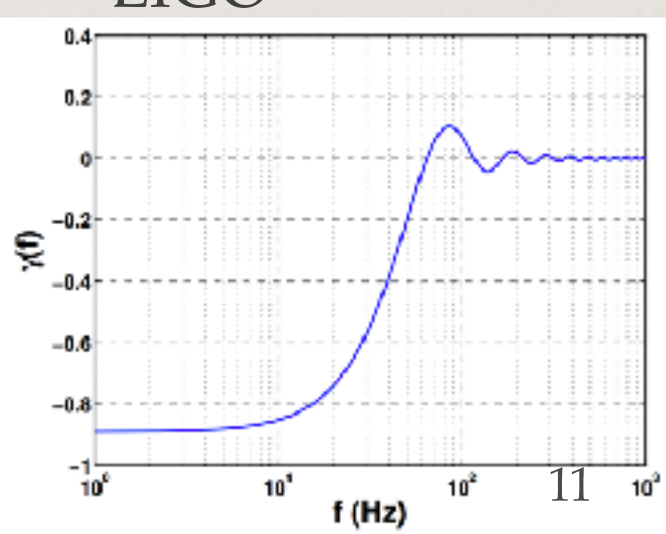
$$\Gamma_{IJ}(f) = \frac{1}{8\pi} \int d\Omega^2 \sum_{+,\times} R_I^{+,\times}(f,\hat{n}) R_J^{*,+\times}(f,\hat{n})$$

For PTA:

$$\Gamma_{IJ}(f) = \frac{1}{3(2\pi f)^2} \chi(\zeta_{IJ})$$

LIGO





In LISA it is also time dependent due motion of the LISA

Anisotropy

Consider Gaussian stationary unpolarized, anisotropic stochastic GW signal. In freq domain the correlation

$$< h_A(f,\hat{n}) h^*_{A'}(f',\hat{n}') >= \frac{1}{4} P(f,\hat{n}) \delta(f-f') \delta_{AA'} \delta^2(\hat{n},\hat{n}')$$

$$S_h(f) = \int d^2\Omega_{\hat{n}}\, P(f,\hat{n})$$

Assume: $P(f,\hat{n}) = H(f) P(\hat{n})$

We can decompose P(n) in spherical harmonics

$$P(\hat{n}) = \sum_{l \geq 0} \sum_{m=-l}^{l} P_{lm} Y_{lm}(\hat{n})$$

Anisotropy

Angular dependence also enters through response (overlap) function

Let us look again at the cross-correlation of two data sets

$$C_{IJ}(t;f) = \frac{2}{\tau}\tilde{d}_I(t;f)\tilde{d}^*_J(t;f)$$

$$< C_{IJ}(t;f) >= H(f)\int d^2\Omega_{\hat{n}}\,\gamma_{IJ}(t;f,\hat{n})P(\hat{n})$$

$$\gamma_{IJ}(t;f,n) = \frac{1}{2}\sum_A R_I^A(t;f,\hat{n})R_J^{*A}(t;f,\hat{n})$$

Overlap reduction function

$$\gamma_{IJ}(t;f,n) = \sum_{l\geq 0}\sum_{m=-l}^{l}\gamma_{lm}(t;f)Y^*_{lm}(\hat{n})$$

$$\gamma_{lm}(t;f) = \gamma_{lm}(0;f)e^{im2\pi t/T_{mod}}$$

$$\boxed{< C_{IJ}(t;f) >= H(f)\sum_{l\geq 0}\sum_{m=-l}^{l}\gamma_{lm}(t;f)P_{lm}}$$

P_{00} Isotropic part

Anisotropy assumptions

Radiometer: set of the point-like objects

$$P(\hat{n}) = P_{\hat{n}_0}\delta^2(\hat{n}, \hat{n}_0)$$

Spherical harmonics decomposition: continuous distribution

$$P(\hat{n}) = \sum_{l\geq 0}\sum_{m=-l}^{l} P_{lm}Y_{lm}(\hat{n}) \qquad \delta\theta \approx \frac{c}{2fD} \approx 180^\circ/l_{max}$$

Non-Gaussian background

Non-Gaussian SGW signal: similar to what we did for the Gaussian SGW -> specify a model and incorporate it into the likelihood

Gaussian:
$$p(h|S_h, M_1) = \frac{1}{(2\pi S_h)^{N/2}} e^{-\frac{\sum_i h_i^2}{2S_h}}$$

Mixed Gaussian
$$p(h|\xi, \alpha, \beta) = \prod_i \left(\xi \frac{1}{2\pi\alpha^2} e^{-\frac{h_i^2}{2\alpha^2}} + (1-\xi) \frac{1}{2\pi\beta^2} e^{-\frac{h_i^2}{2\beta^2}} \right)$$

Superposition of large number of deterministic signals

$$p(h|\vec{\theta}, M_1) = \delta(h - h(\vec{\theta}))$$
$$h(\vec{\theta}) = \sum_{k=1}^{M} A_k h_d(t - t_k, \vec{\lambda}_d)$$

Summary

- Need to have model for the noise (Gaussian?)
- Need to decide on the model of SGW signal (isotropy / anisotropy, Gaussianity)
- Based on the above: write likelihood
- Define priors (in case of the Bayesian analysis)
- Perform sampling of the posterior (Markov chain mote carlo, nested sampling, combined methods).

(Power) SNR Assumes weak signal limit, isotropic, sky and polarisation average

$$\rho^2 = \frac{2}{T} \int df \sum_{I \geq 1} \sum_{J > I} \frac{\Gamma^2_{IJ}(f) S^2_h(f)}{S_{n_I} S_{n_j}}$$