Equilibration at Weak Coupling

Guy D. Moore, Aleksi Kurkela, Mark York, Egang Lu, 1107.5050,1108.4684,1207.1663,1401.3751,1405.6318

- Motivation: cosmology and heavy ion collisions
- The issues:
  - ∗ High vs Low Occupancy
  - ∗ Gauge vs Scalar/Yukawa theory
  - ∗ Isotropic vs Anisotropic
- Physics: Soft scattering, collinear splitting, plasma instabilities
- Some qualitative, some quantitative results

Motivation

Where you may meet gauge theories far from equilibrium:

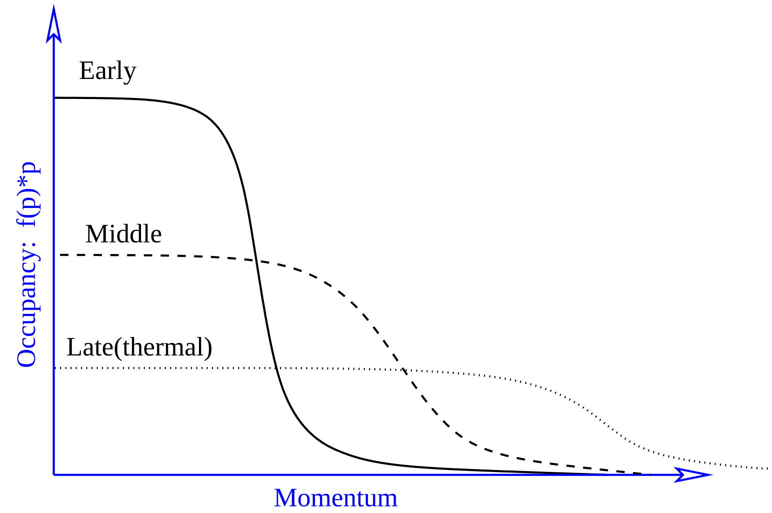
- Cosmology: (p)reheating (decay products, parametric resonance....)
- Cosmology: phase transitions, *eg.*, electroweak
- Heavy ion collisions (only weakly coupled in ultra-high energy limit)

I will only consider:

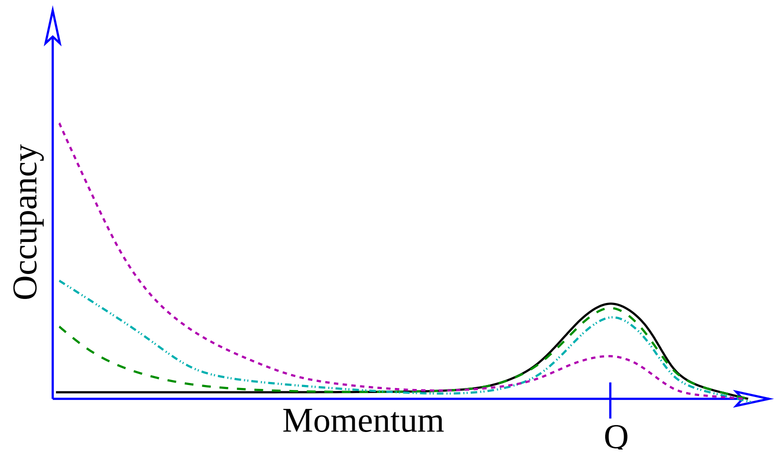
- weak coupling $\alpha \ll 1$ (Cosmology: OK. HIC: ??!)
- Mostly parametric estimates arXiv:1107.5050 – 55pp, 183eq, 15fig, 619 $\sim$'s!
- homogeneous systems Not as bad as it sounds

High versus Low Occupancy

High occupancy: need to reduce part. number, raise energy. Scattering, number-change both relevant.

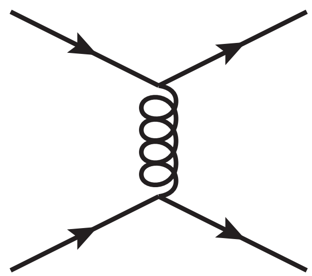
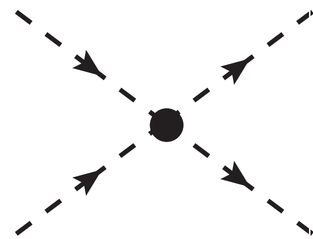


Low occupancy: need to split particles up into more, lower energy. Key physics is number-changing processes (LPM...)

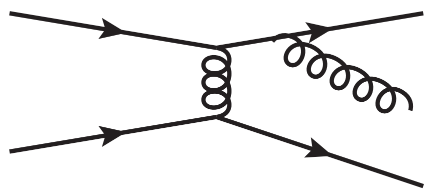
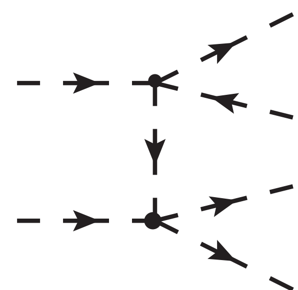


Gauge vs scalar theories

Scalar thy=favorite toy model (2PI methods...) **BUT**

Elastic scatt:  **versus** 

t-channel: Gauge thy has large $\mathcal{M}^2$ at small t

Inelastic scatt:  **Versus** 

Gauge process *soft* **and** *collinear* enhanced

Scalar: super slow. Gauge: fast as large-angle elastic!

Isotropic systems

If typical occupancies $f(k) < 1/\alpha$, weak coupling implies *quasi-particles*, evolve under a Boltzmann equation.

Need *Effective Kinetic Theory* Arnold GM Yaffe V, hep-ph/0209353 treating:

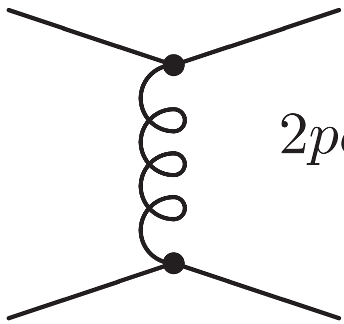
- Elastic scattering processes
- Inelastic number-changing "effective" processes

And we need a *quantitative numerical algorithm*

Kurkela Lu GM York 1401.3751, Kurkela Lu 1405.6318, Lu GM in prep.

which implements the kinetic thy.

Elastic scattering



$$2p\partial_t f(p) \;=\; \int_{k,p',k'} |\mathcal{M}|^2 \Big(f(p)f(k)[1{+}f(p')][1{+}f(k')] \\ - f(p')f(k')[1{+}f(p)][1{+}f(k)] \Big)$$

Naively $|\mathcal{M}|^2$ diverges as $1/q^4$, a problem.

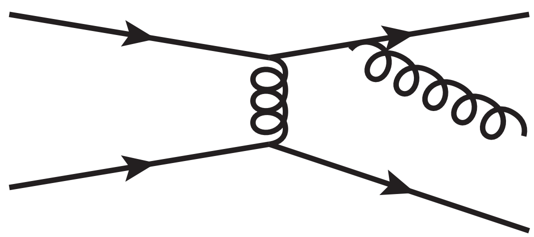
Screening corrects this. At leading-order, can use

$$|\mathcal{M}|^2 \propto \frac{1}{q^4} \Rightarrow \frac{1}{(q^2+\Pi(q,\omega,m_{\mathrm{D}}))^2} \quad \Longrightarrow \quad \frac{1}{(q^2+m^2)^2}$$

for carefully chosen $m = e^{5/6} 2^{-3/2} m_{\mathrm{D}}$ Kurkela Lu GM York.

Inelastic processes

Soft scattering can cause radiation/absorption!

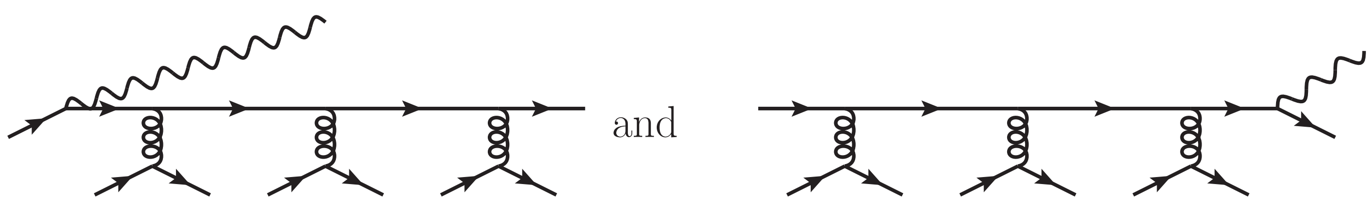


$$\Gamma \sim \alpha[1+f(r)]\Gamma_{\rm soft}$$

For over-occupied case, *same order* as $\Gamma_{\rm large-angle}$.

For under-occupied, actually *faster!* Also, $\Gamma_{p\to k} \propto 1/k$.

Bad news: interference between these:



non-negligible. Have to work harder to get inelastic rate!

Boltzmann equation: Expression

Skipping details, effective Boltzmann equation is: (AMY5)

$$
\begin{aligned}
\frac{\partial f(p,t)}{\partial t} &= -\mathcal{C}_{2\leftrightarrow 2}[f(p,t)] - \mathcal{C}_{1\leftrightarrow 2}[f(p,t)]\,, \\
\mathcal{C}_{2\leftrightarrow 2}[f_p] &= \frac{1}{2\nu_g} \int \frac{d^3k}{(2\pi^3)} \frac{d^3p'}{(2\pi^3)} \frac{d^3k'}{(2\pi^3)} \frac{\left|\overline{\mathcal{M}}^2_{pk;p'k'}\right|}{2p2k2p'2k'} (2\pi)^4 \delta^4(p{+}k{-}p'{-}k') \times \\
&\qquad \Big(f_p f_k [1+f_{p'}][1+f_{k'}] - [1+f_p][1+f_k] f_{p'} f_{k'} \Big)\,, \\
\mathcal{C}_{1\leftrightarrow 2}[f_p] &= \frac{(2\pi)^3}{p^2} \int_0^{\frac{p}{2}} dk\, \gamma_{\mathrm{split}}(p;k,p{-}k) \Big(f_p f_{p-k} + f_p f_k - f_k f_{p-k} \Big) \\
&\quad + \frac{(2\pi)^3}{p^2} \int_0^{\infty} dk\, \gamma_{\mathrm{split}}(p{+}k;p,k) \Big(f_p f_k - f_p f_{p+k} - f_k f_{p+k} \Big)
\end{aligned}
$$

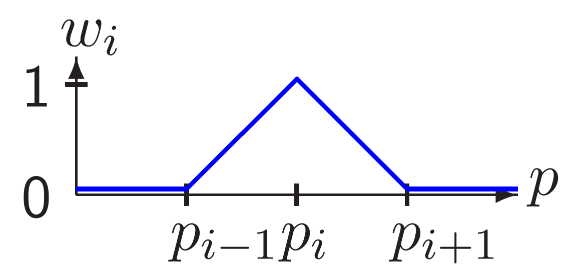
$\gamma_{\mathrm{split}}(p;k,p-k)$: effective rate for medium scatt to induce $p \to k, (p{-}k)$

expression is horrible but known, do-able numerical problem

Boltzmann: Algorithm

I want a discretization based tightly on continuum Boltz. Eq.
Choose discrete set of momenta p_i. Also define "wedge-function"

$$w_i(p) \equiv \begin{cases} \frac{p-p_{i-1}}{p_i-p_{i-1}}\,, & p_{i-1} < p < p_i \\ \frac{p_{i+1}-p}{p_{i+1}-p_i}\,, & p_i < p < p_{i+1} \\ 0 & p < p_{i-1} \text{ or } p > p_{i+1} \end{cases} \quad =$$



Number of particles with momentum "near" p_i is

$$n_i \equiv \int \frac{d^3p}{(2\pi)^3} f(p) w_i(p) \quad \text{so} \;\; n = \sum_i n_i\,, \quad \varepsilon = \sum_i p_i n_i\,.$$

Base discrete version on solving for n_i.

Algorithm concluded

What is $\partial_t n_i$??

$$\begin{aligned}
\partial_t n_i &= \int_p w_i(p)\, \partial_t f(p) \\
&= \int_{p,k,p',k'} \frac{|\mathcal{M}|^2}{\ldots} \delta^4(\ldots)(f_p f_k (1 \pm f_{p'})(1 \pm f_{k'}) - \ldots) w_i(p) \;+\; \Big(1 \leftrightarrow 2\Big) \\
&= \int_{p,k,p',k'} \text{(collision rate)}\; \frac{w_i(p)+w_i(k)-w_i(p')-w_i(k')}{4} \;+\; \Big(1 \leftrightarrow 2\Big)
\end{aligned}$$

Each p, k, p', k' has only 2 nonzero w_i's. Do the integrals once (all-but-3 closed-form), learn *all* $\partial_t n_i$'s! Method IR safe. *exact* energy cons., $\mathcal{C}_{2\leftrightarrow 2}$ *exactly* converves particle number.

Only one approximation needed: assume $f(p)$ inside $\int$ can be interpolated from values of n_i.

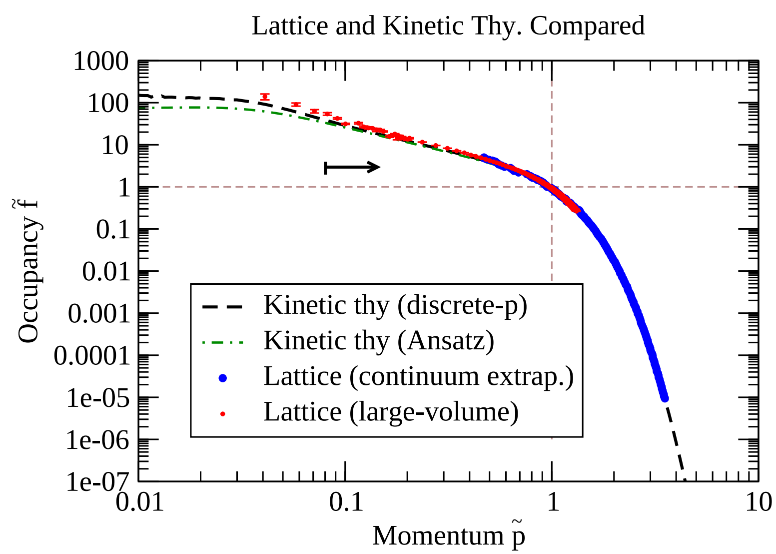
Results: Over-occupied

Early stages: classical thy.

Solveable on lattice.

Kinetic thy fits like a glove

Kurkela Lu GM York



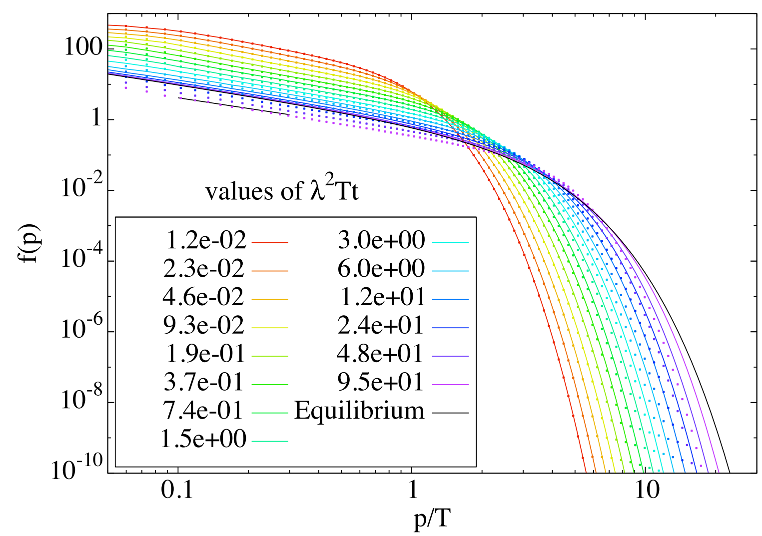
Late stages (pure glue):

quantitative description of

approach to equilibrium.

Can now treat quarks too

Kurkela Lu, Lu GM in prep



Results: Underoccupied

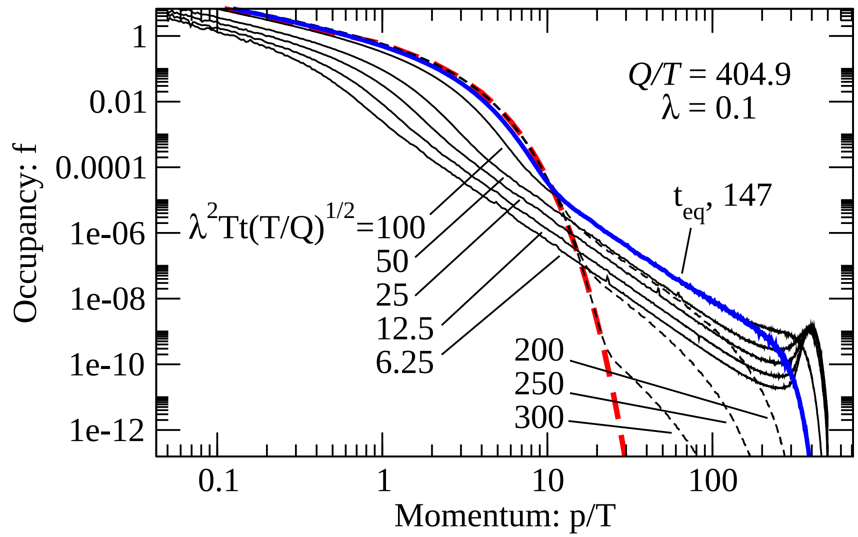
Bottom-up behavior

(form soft bath first)

Equil. Time

$t \sim \alpha^{-2} Q^{1/2} T^{-3/2}$

Quantitative results



Kurkela Lu

Scalar thy completely different: slow kinetic equil.
$t_{\rm kin} \sim \lambda^{-2} Q^3 T^{-4}$, to $f(p) = 1/(e^{(E+\mu)/T} - 1)$ Then erase chemical potential much slower, $t_{\rm full\ therm} \sim \lambda^{-4} Q^3 T^{-4}$
Different powers of coupling and of Q.

Anisotropic Systems

First complication: need $f(p,\theta)$ not $f(p)$.

Need keep track of energy *and* angle change.

Screening:

$$G_T^{\mu\nu}(Q) = \frac{P_T^{\mu\nu}}{Q^2 + \Pi_T}\,, \qquad G_L^{\mu\nu}(Q) = \frac{P_L^{\mu\nu}}{Q^2 + \Pi_L}\,.$$

where P_T, P_L trans. and longit. projectors

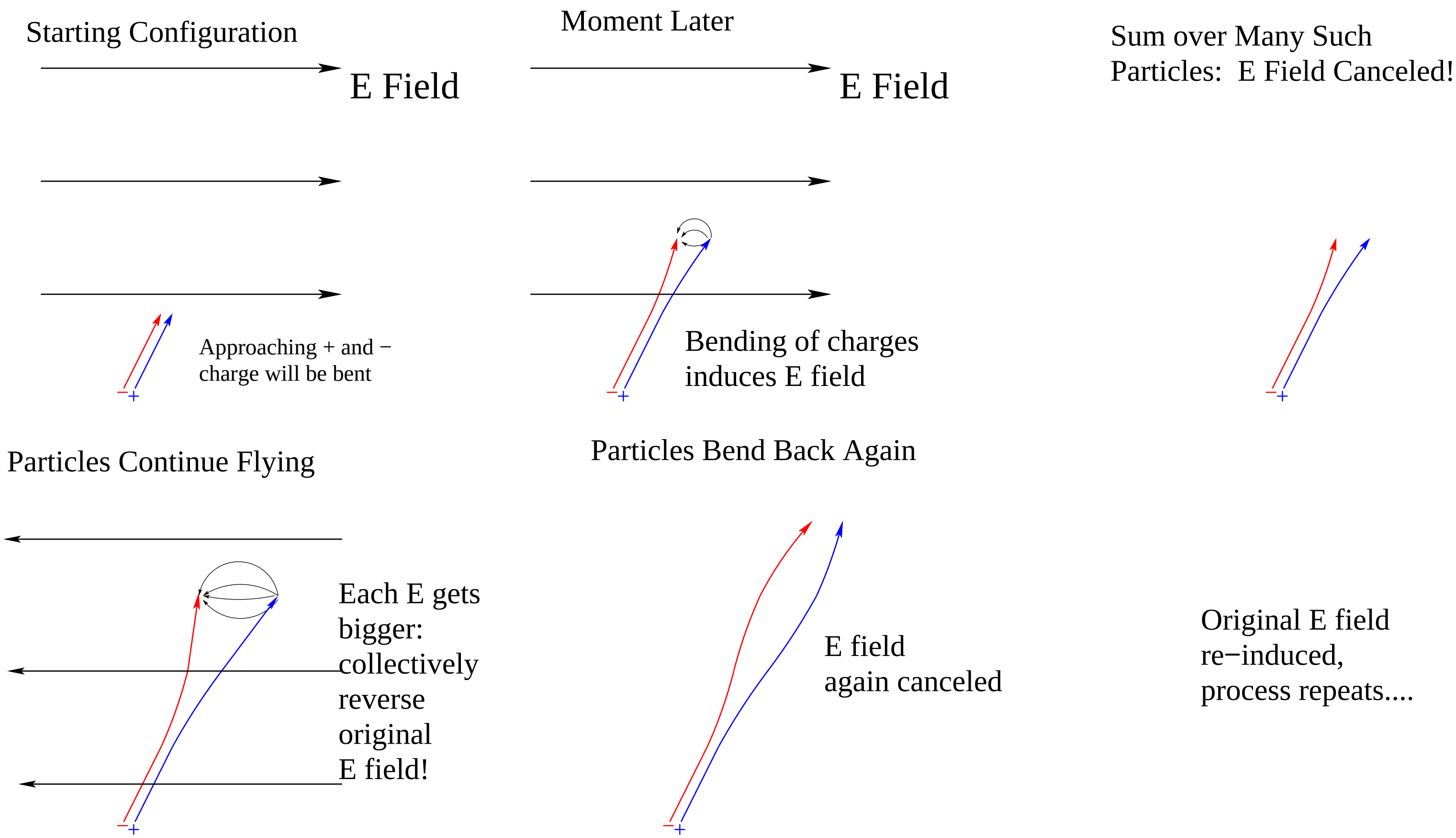
Π only matters at Small $Q \;\Rightarrow\;$ momentum diffusion!

$f(p)$: Only energy change: can make "cheap" approx

$\Pi = (Q^2/q^2)m^2$, choose m^2 to get E-diffusion right.

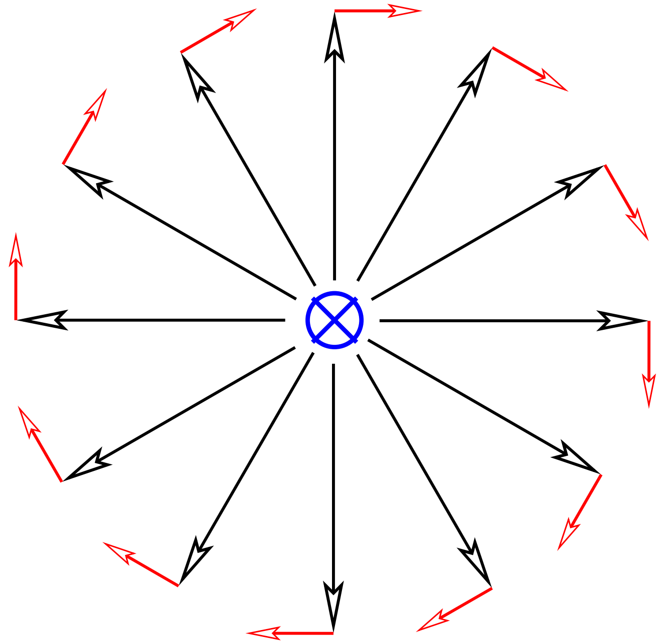
Angle-change: two things to get right. Think harder.

Screening of E fields



Any $G^{\mu\nu}(Q)$ with E-fields acts “massive” (screened)

Screening of B fields



B-field quite different:
for B out-of-board,
isotropic dist. just gets
rotated – not changed.

No current induced – plasma doesn't affect B.

Isotropic: E, but not B screened.
Therefore $\Pi_T(q^0 = 0, q) = 0$ (B-only).

Put head down and compute?

Simply compute Π_L and Π_T for anisotropic system?
$\Pi(q^0, q, \theta_q)$ angle-dependent. Also, more tensor structures.
Result: $\Pi > 0$ for most angles & tensors but < 0 for some

Mrówczyński 1988 *etc*, Arnold Lenaghan GM hep-ph/0307325, Romatschke Strickland hep-ph/0304092

Matrix element involves

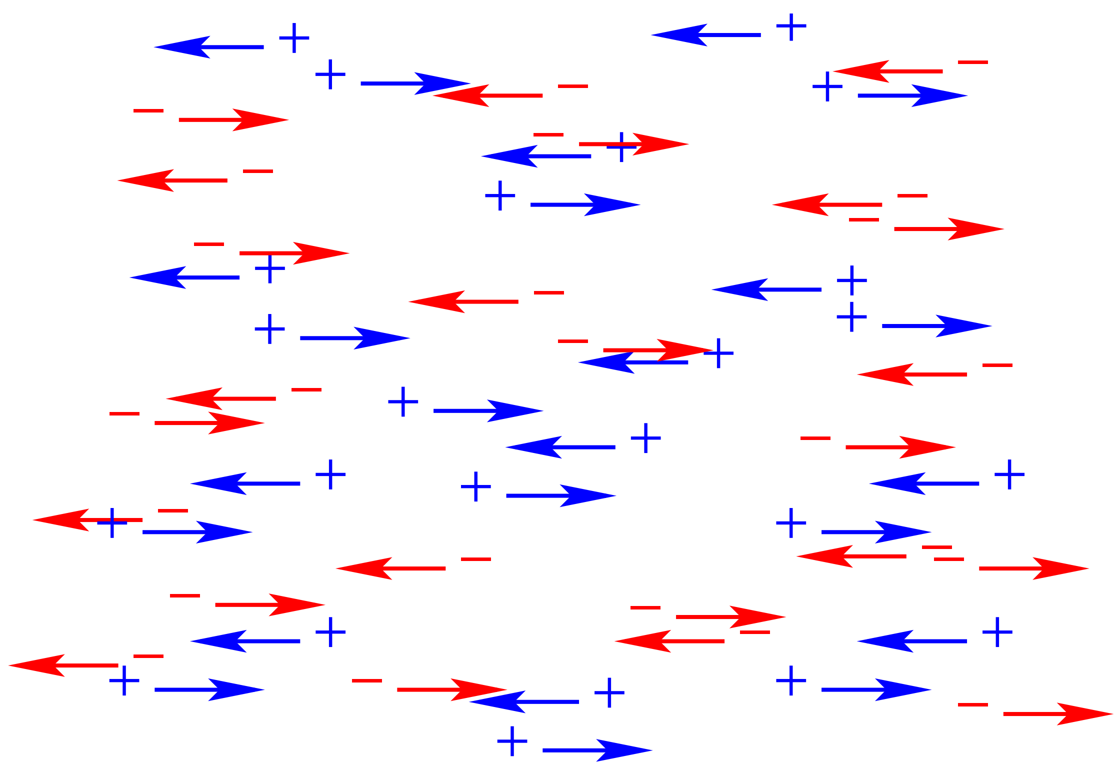
$$\mathcal{M} \propto G \sim \frac{1}{q^2 + \Pi}$$

can now have zero in denom. at finite q^2.
Try using this $\Longrightarrow$ get infinite scatt rates!

We will need to understand what this means!

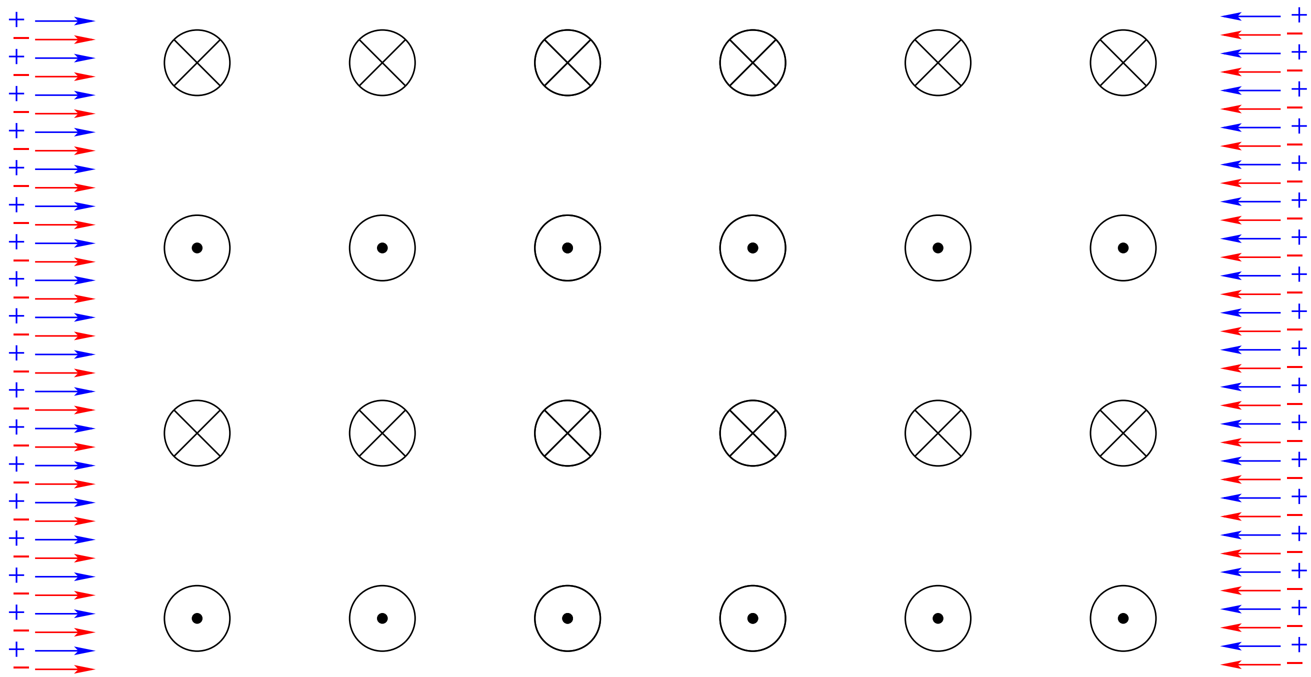
Anisotropic medium: Instabilities!

For illustration: Consider maximum anisotropy $\rightarrow$ all particles move only in z direction:



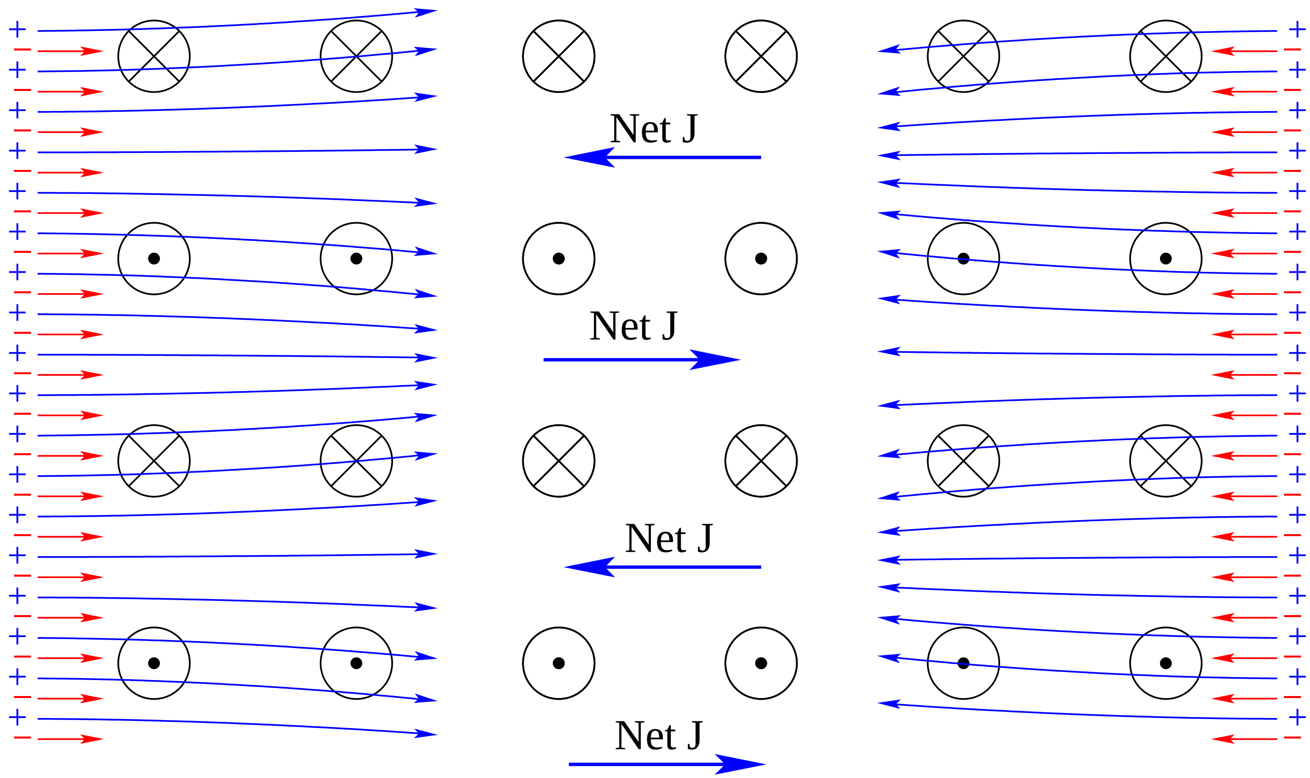
Magnetic field growth!

Consider the effects of a seed magnetic field $\hat{B} \cdot \hat{p} = 0$ and $\hat{k} \cdot \hat{p} = 0$



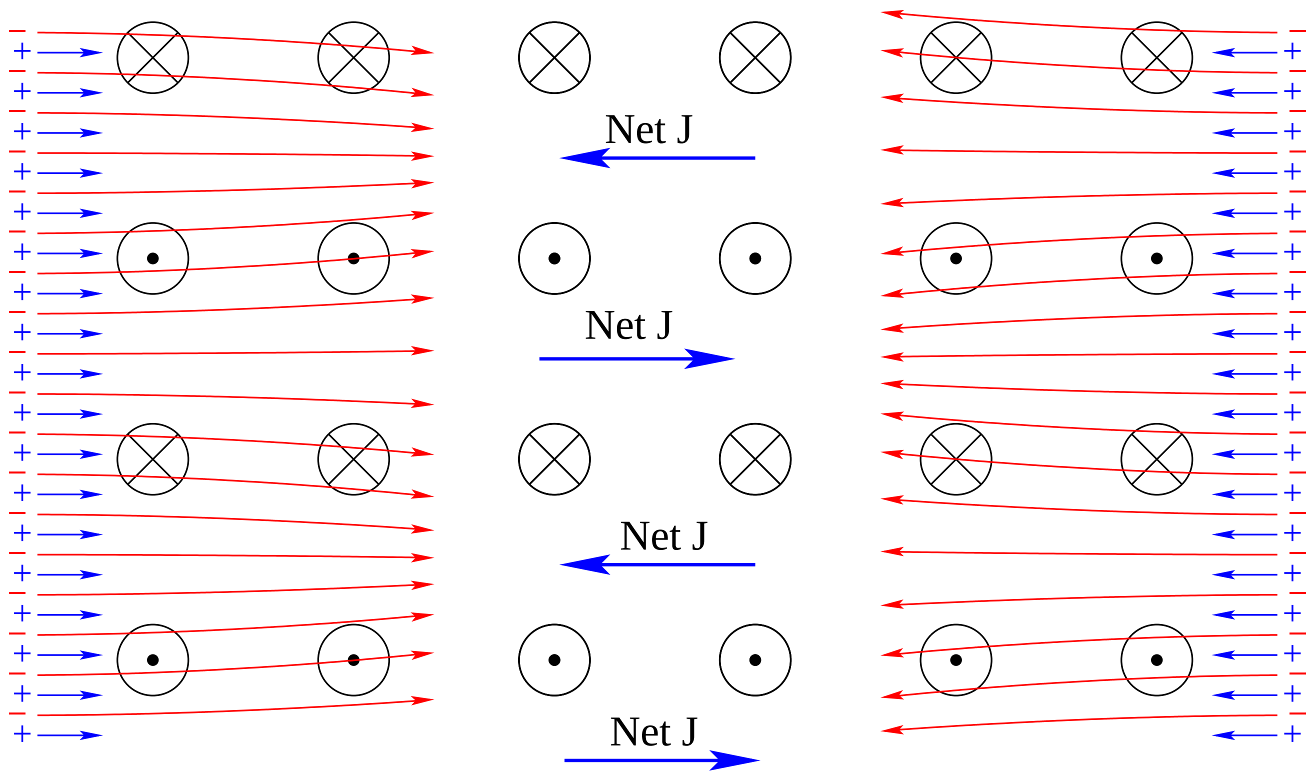
How do the particles deflect?

Positive charges:



No net ρ. Net current is induced as indicated.

Negative charges: same-sign current contribution



Induced B *adds* to seed B. Exponential Weibel instability.

Guesstimate of growth rate:

Force on particle $\mathbf{F} \sim g\mathbf{B}$. Velocity change $\mathbf{v} \sim \mathbf{F}t/p$

Deflection: $\Delta x \sim \mathbf{v}t \sim \mathbf{F}t^2/p \sim gBt^2/p$

Concentration: $k\Delta x$. Current per particle: $g(k\Delta x)$.

Current: $J \sim \int d^3p f(p)\, g(k\Delta x) \sim \int \frac{d^3p}{p} f(p) g^2\, k\mathbf{B}t^2$

That is, $J \sim m^2 t^2 k\mathbf{B}$

Current matters when $J \sim \nabla \times B \sim kB$, which is $m^2t^2 \sim 1$.

Growth rate must be $\Gamma \sim m$.

Growth occurs *iff* particles stay in same-sign B for $t \gtrsim 1/m$.
(Otherwise J never builds up.)

Weak anisotropy

Define angular distribution $\Omega(\mathbf{v})$:

$$\Omega(\mathbf{v}) \equiv \int \frac{d^3\mathbf{p}}{E} f(\mathbf{p}) \delta(\hat{\mathbf{p}} - \mathbf{v})$$

Weak anisotropy means $\Omega(\mathbf{v}) = \Omega + \epsilon Y_{20} + \ldots$

Large isotropic part plus small anisotropic extra.

Only aniso. bit causes instability. $m_{\text{eff}}^2 \sim \epsilon m^2$.

Particles must be in same-sign B for $t \sim 1/m\sqrt{\epsilon}$.

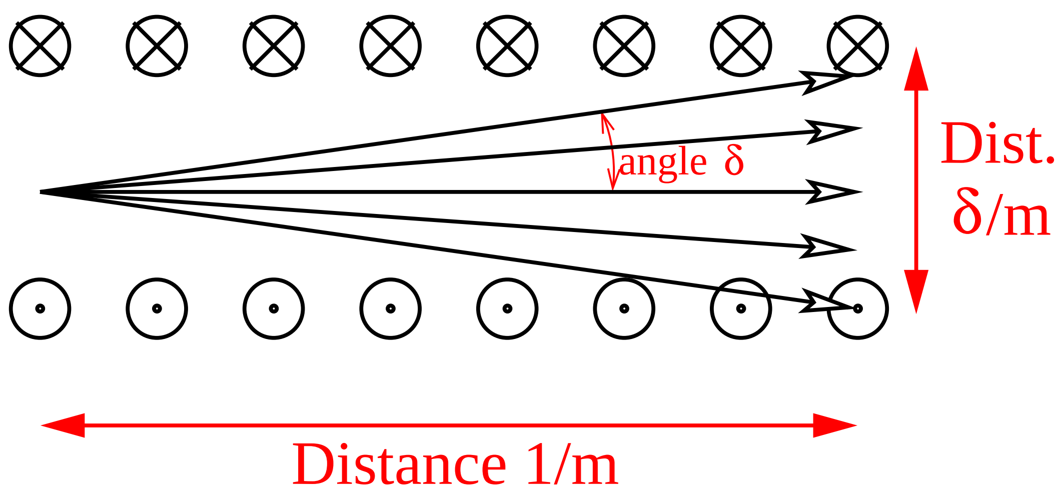
Hence unstable $\mathbf{k}$ have $|\mathbf{k}| \sim \sqrt{\epsilon} m$.

Growth rate $\Gamma \sim \sqrt{\epsilon} m$. Actually $\epsilon^{\frac{3}{2}} m$ due to E-screening.

Strong anisotropy

What happens when $\Omega(\mathbf{v})$ peaked in narrow angle range?

$\Omega(\mathbf{v})$ small unless $|v_z| < \delta \ll 1$?

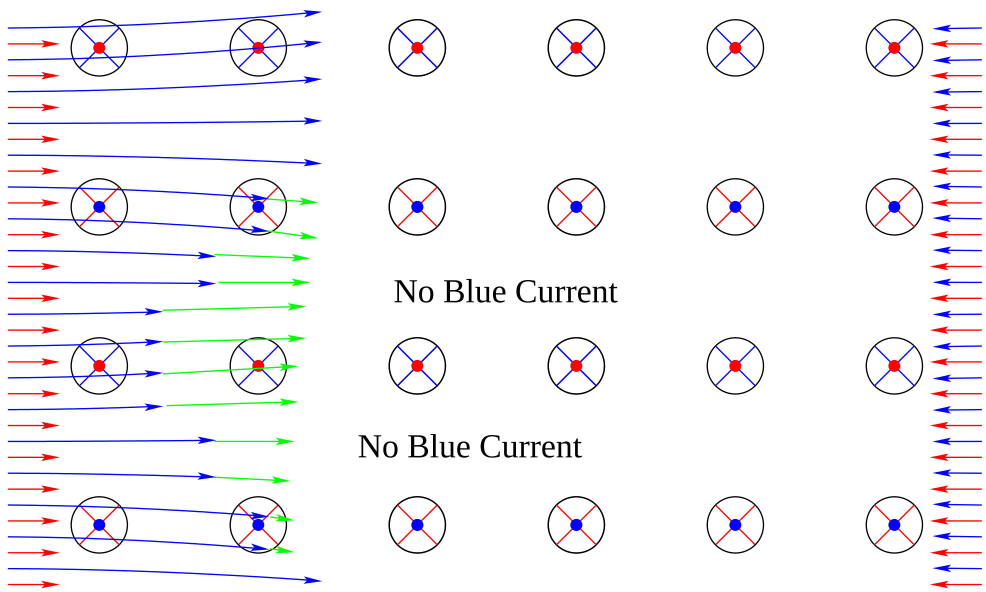


Narrow z-spacing of B's (large k_z) still allowed!

Instability for $\mathbf{k} \sim (m, m, m/\delta)$, growth $\Gamma \sim m$.

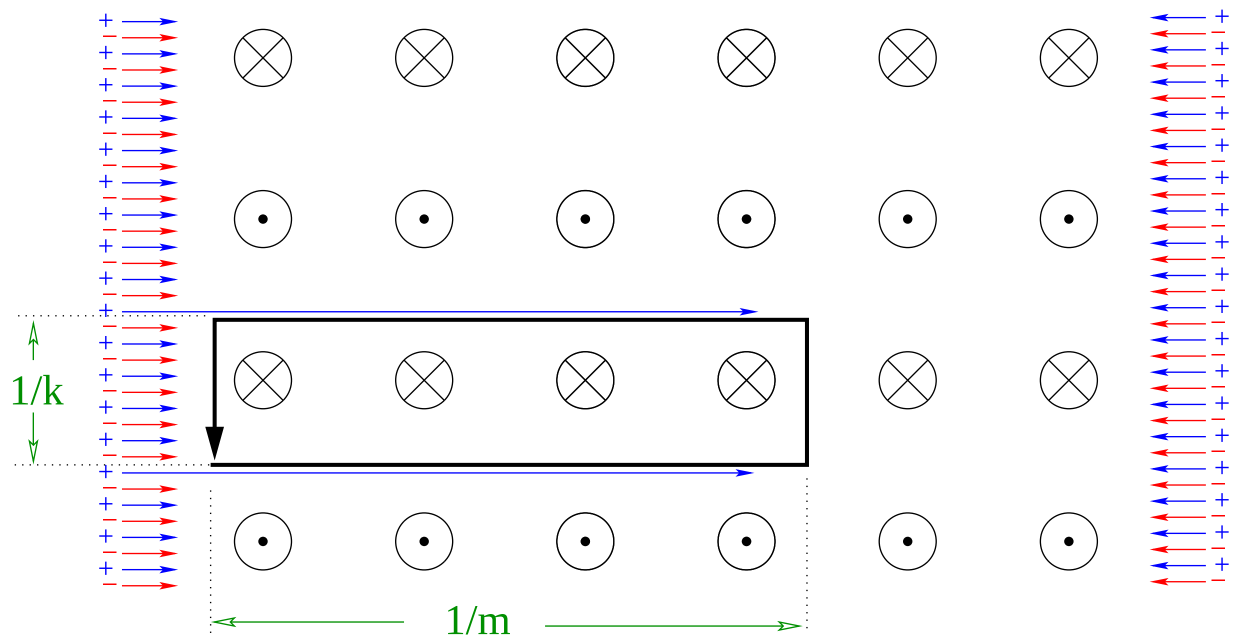
What limits B field growth?

Color randomization!



B growing in all colors, many k at once. Large B: Wilson lines so $\neq \mathbf{1}$ that color rotation happens. Growth cut-off if color-coherence shorter than $1/m$ $[1/(m\sqrt{\epsilon})$ weak-aniso]

Proper gauge invariant version

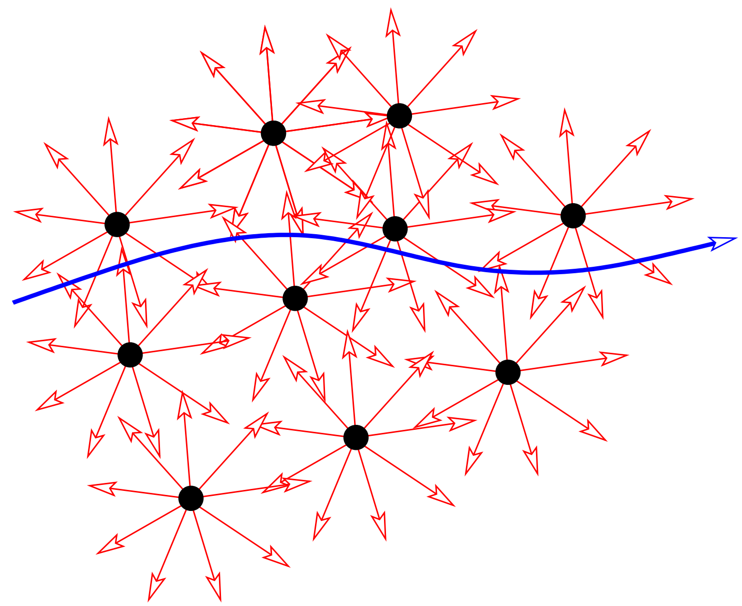


Wilson loop must contain $\mathcal{O}(1)$ phase. Requires $B \sim km/g$.

Weak aniso: $B \sim \epsilon m^2/g$. Strong aniso: $B \sim m^2/\delta$.

Current understanding

Soft-exchange scatt. is deflection in the E, B fields of the other charges in the plasma



Normal isotropic plasma: same as E, B having occupancy $f(q) \sim T_*/q$. Scatt rate $\propto \int_q (\ldots) f(q)$.

Anisotropic plasma: in the unstable "band," occupancy is instead nonpert. large, $f(q) \sim 1/g^2$

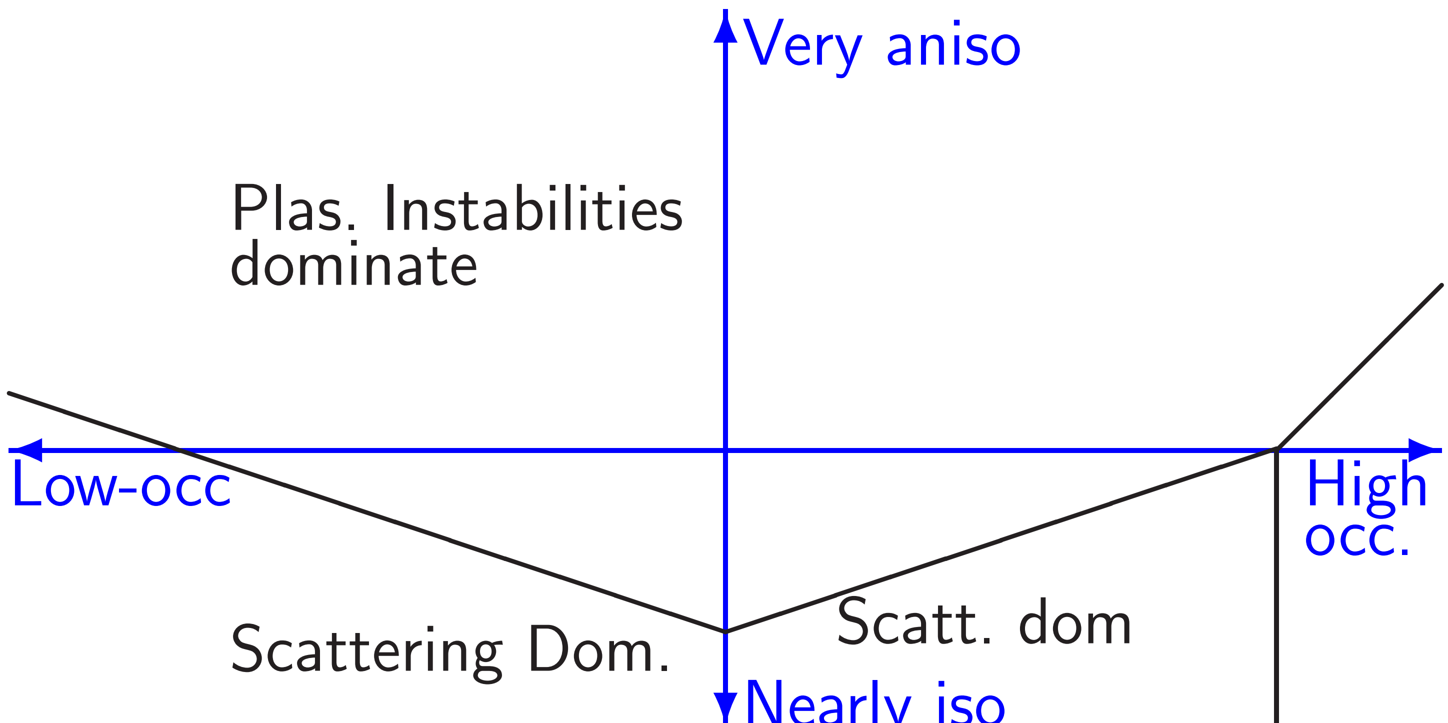
Represents large enhancement **if** T_*/q is perturbative.

Occupancy-Anisotropy Plane

Consider system with one characteristic p-scale Q

Strong aniso: θ-range is $\delta \equiv \alpha^{d}$. Weak: $\delta f(\theta)/f \equiv \epsilon = \alpha^{-d}$.

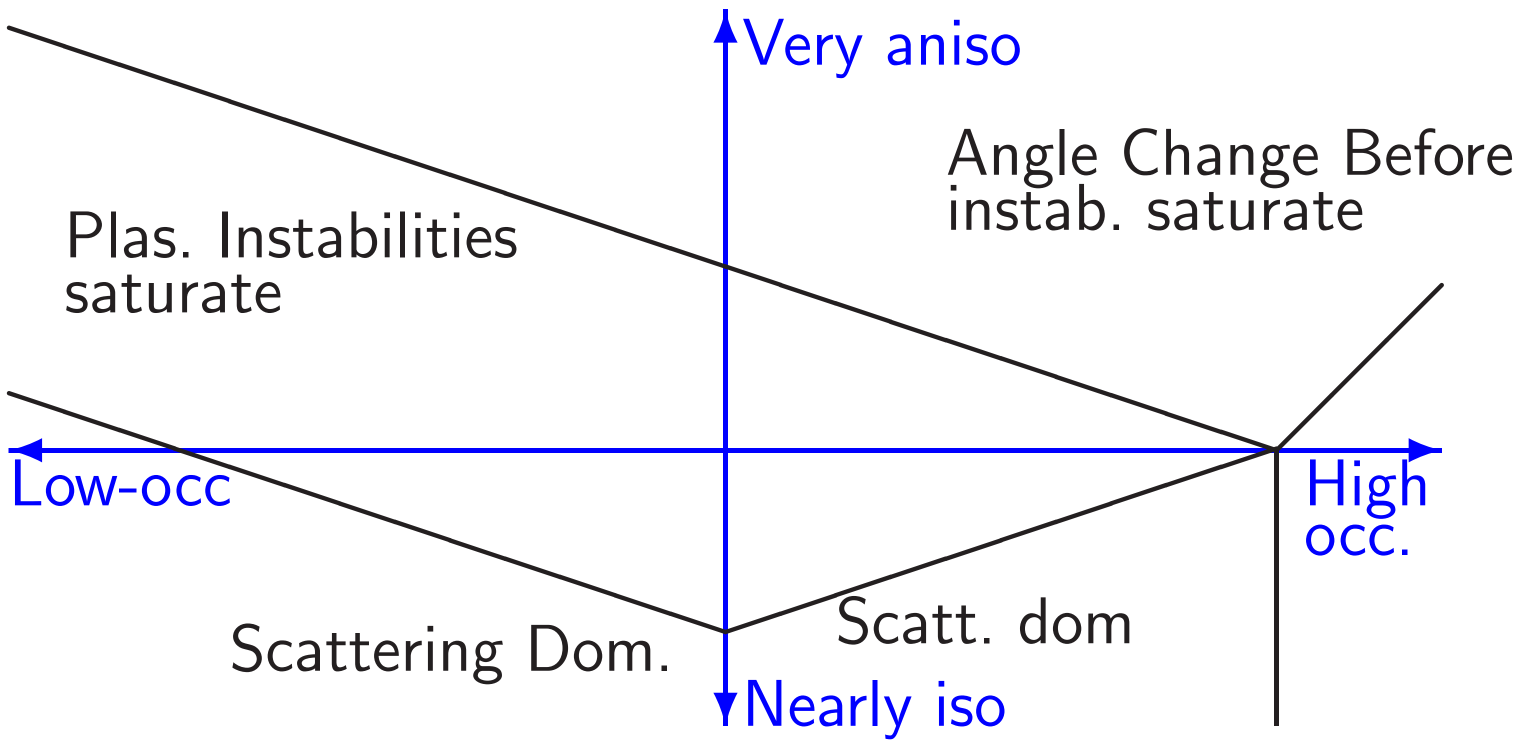
Call typical occupancy $f(p \sim Q, p_z = 0) \sim \alpha^{-c}$.



Angle change

Angle change matters when $\Delta\theta > \delta \; (d > 0) \; or > 1 \; (d < 0)$

Can happen before or after plas. inst. finish growing:



Radiated Daughters

Plas. instabilities raise rate of soft radiation.

Radiated daughters are born anisotropic.

Can have their own plasma instabilities! (¡Ay Caramba!)

Driven to isotropy by plas. instabilities, scattering, their own plas. instabilities.

Become important when they dominate scattering – typically by having their own plasma instabilities.

Merging is anisotropic and can also be important!

My complication had a complication

Physics sensitive to occupancy and anisotropy:

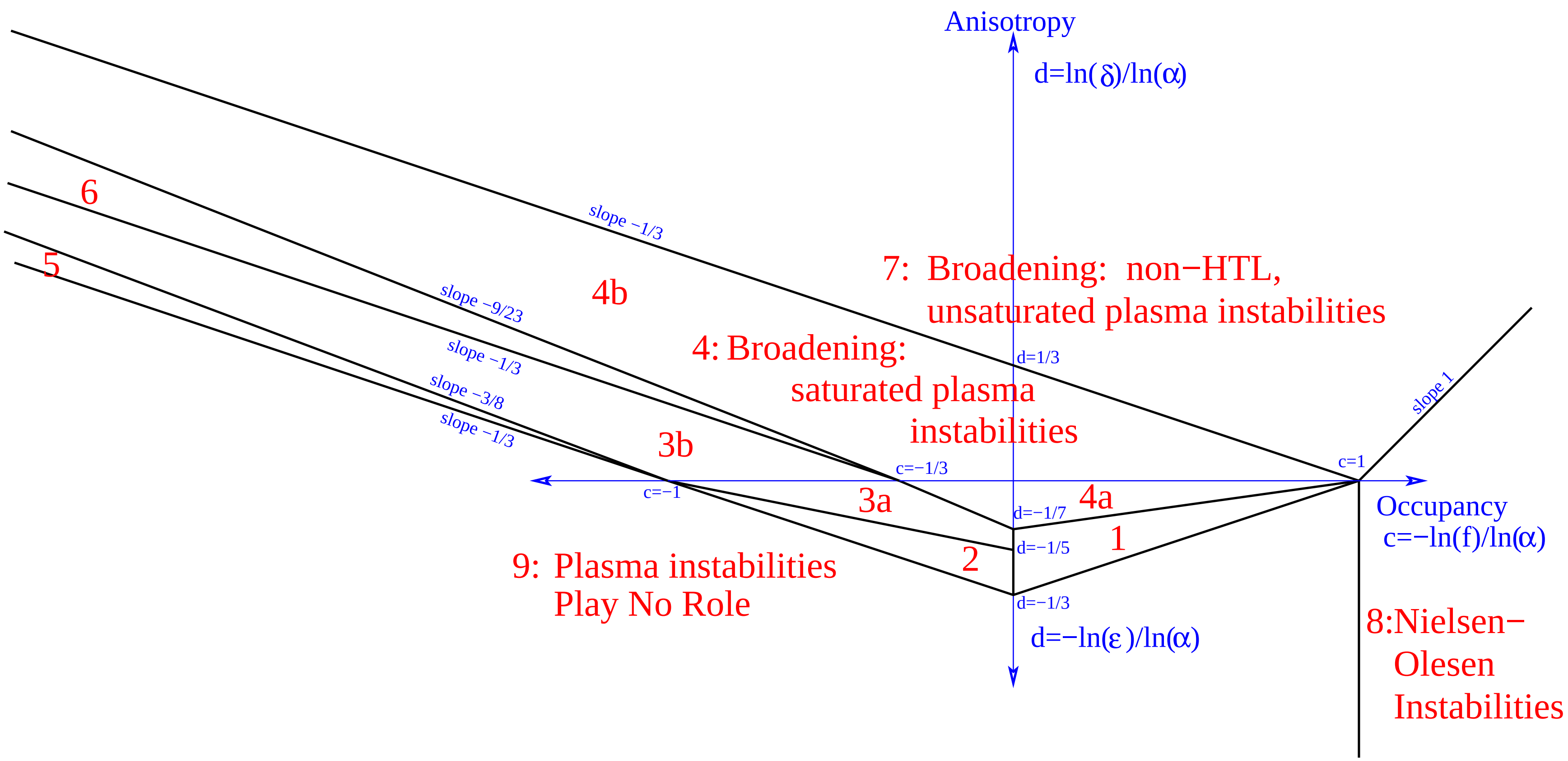
- More anisotropic: plasma instabilities more effective.
- Low occupancy: easier for daughters to become important.

When considering daughters, 3 scales which evolve with time:

- Scale $k_{\rm re-join}$ where daughters so numerous that re-merging onto hard modes occurs. Scales with time as $t^{2/5}$
- Scale $k_{\rm iso}$ where daughters' directions randomized: scales as $t^{1/2}$
- Scale $k_{\rm split}$ where daughters split again into lower-momentum "grand-daughters". Scales as t^1.

30 pages, 349 $\sim$'s later....

Anisotropic Case: Summary



1: merging dominated. 2,3: noneq. daughters. 5: daughters before instabilities saturate. 6: almost-thermal daughters. 10(not shown): thermal daughters

Practical issues

Do plas. instabilities really matter?
Seems to depend on $f(q) \sim T_*/q \ll 1/g^2$, same as there being large separation between $m_{\rm D}$ and g^2T scales.

Lattics: coupling has to be really small before this separation exists.

Suggests (Much more work needed!) that plas instabilities may be a red herring.

Conclusions and Questions

- Isotropic: physics of elastic and inelastic scattering
- Practical algorithm, quantitative results!
- Anisotropic: plasma instabilities drive dynamics.
  Daughters cause own instabilities Wheels within Wheels

To do:

- Understand when *in practice* instabilities really matter
- What estimates can I make quantitative for aniso. case?