GRAVITATIONAL WAVES

by Chiara Caprini

SUMMARY:

1) Definition of GWs in linearised theory, Lorenz and transverse - traceless gauge, polarisation modes

2) Interaction of GWs with test masses and basic description of the principles of GW detection

3) Energy momentum tensor of GWs

4) Generation of GWs in linearised theory: low - velocity expansion, quadrupole formula, GW signal by binary systems

5) Inspiral of compact binaries in linearised theory

6) Cosmological application: Standard Sirens

References

(obis)

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GRAVITATIONAL WAVES

SOME FACTS:

* GWs emerge naturally in GR, because GR extends Newtonian theory and makes it compatible with SR ⇒ it renders Newtonian theory CAUSAL. From the requirement of causality one has that a change in the gravitational potential at the position of the source must be communicated to a distant observer no faster than the speed of light: therefore, there must be some form of radiation that carries information at the speed of light: GWs

* The source of GWs are accelerated masses (as in EM: accelerated charges)

* the gravitational interaction is weak: it wins over all other interactions in the universe at large distances only because it is always attractive. In fact, it took 50 years from the first attempts of building GW detectors before a direct detection was made the last year finally. For the same reason, one cannot hope to produce detectable GWs on Earth: to get a sufficiently strong signal one needs very big

masses moving at high velocities: astrophysical objects (or the very early universe: very energetic processes). ②

* The first experimental evidence of the existence of GWs arised from indirect measurements: the observation of binaries of neutron stars. As the two stars inspiral towards each other they emit GWs; this GW emission is strong enough that it back-reacts on the dynamics of the binary system on a time-scale short enough to be observable. The emission of GWs carries away energy and angular momentum from the system, reducing the size of the orbit. This effect has been measured in the Hulse-Taylor binary pulsar (Nobel prize in 1993): a system of a pulsar orbiting a NS. The change in the duration of the pulses is on a daily basis, so by monitoring the system for a long time it was possible to measure the shrinking of the radius of the orbit due to the emission of GWs. This measurement matches very well the GR prediction (figure)

* On the 14/9/2015 there has been the first direct detection of GWs from the LIGO interferometers: two interferometers based in the USA (Washington and Louisiana states). Since, other "two" detections (one of the two less certain) have been announced.

The first detection, with the highest SNR, was the GWs emitted by a binary of two Black holes in the last phases of the inspiral just before they merged.

$M_1 = 36 \pm 5\, M_\odot$

$M_2 = 29 \pm 4\, M_\odot$

$R_s = 210$ km

$z = 0.09 \pm 0.04$

(show figure)

* This has opened the era of GW astronomy, in which we will use GWs together with EM radiation to probe the universe.

GWs can carry informations on the states of the universe from z much higher than EM radiation: because of the weakness of the gravitational interaction, GWs are decoupled since the Planck epoch.

GRAVITATIONAL WAVES IN LINEARIZED THEORY

GWs are derived in the context of linearised theory:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1$$

we expand around the Minkowski metric. Here we have picked a reference frame, and we have therefore broken the invariance of GR under coordinate transformation. The values of $h_{\mu\nu}$ depend on the coordinate choice; we have chosen a frame in which the above can be written, $|h_{\mu\nu}| \ll 1$.

Even after having chosen this frame, there remains a residual freedom in the choice of coordinates:

coordinate transformation

$$x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu(x) \qquad \text{with} \quad |\partial_\mu \xi^\nu| = \mathcal{O}(h)$$

how does the metric change?

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial x^\sigma}{\partial \tilde{x}^\nu} g_{\rho\sigma}(x) \qquad \text{gives to lowest order}$$

$$h_{\mu\nu}(x) \to \tilde{h}_{\mu\nu}(\tilde{x}) = h_{\mu\nu}(x) - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu)$$

by the requirement $|\partial_\mu \xi^\nu| = \mathcal{O}(h)$ we get that $|\tilde{h}_{\mu\nu}| \ll 1$: therefore, slowly varying coordinate transformations are a symmetry of the linearised theory because it remains a linearised theory around Minkowsky.

The christoffel symbols in linearised theory are:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right) =$$

$$= \frac{1}{2} \left(\partial_\mu h^{\rho}{}_{\nu} + \partial_\nu h^{\rho}{}_{\mu} - \partial^{\rho} h_{\mu\nu} \right) + \mathcal{O}(h^2)$$

the Riemann tensor in linearised theory is:

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\rho} + \Gamma^{\mu}_{\alpha\rho} \Gamma^{\alpha}_{\nu\sigma} - \Gamma^{\mu}_{\alpha\sigma} \Gamma^{\alpha}_{\nu\rho}$$

$$= \frac{1}{2} \left(\partial_\nu \partial_\rho h^{\mu}{}_{\sigma} + \partial^{\mu} \partial_\sigma h_{\nu\rho} - \partial^{\mu} \partial_\rho h_{\nu\sigma} - \partial_\nu \partial_\sigma h^{\mu}{}_{\rho} \right) + \mathcal{O}(h^2)$$

Note that the Riemann tensor is INVARIANT under the coordinate transformation we consider.

(in quantities that are first order in h we raise and lower indices with $\eta_{\mu\nu}$)

EQUATIONS OF MOTION IN LINEARISED THEORY:

From the Riemann tensor we can find the Einstein tensor, plug it into Einstein eqs. and find how they look like in linearised theory.

We first introduce the TRACE-REVERSED PERTURBATION

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h \qquad \text{with } h = h_{\mu\nu} \eta^{\mu\nu}$$

in terms of which Einstein eqs take the form:

$$\Box \bar{h}_{\mu\nu} + \eta_{\mu\nu} \partial^\rho \partial^\sigma \bar{h}_{\rho\sigma} - \partial^\rho \partial_\nu \bar{h}_{\mu\rho} - \partial^\rho \partial_\nu \bar{h}_{\nu\rho} =$$

$$= -\frac{16\pi G}{c^4} T_{\mu\nu}$$

This equation would simplify a lot if we could pick a frame in which

$$\boxed{\partial^\nu \bar{h}_{\mu\nu} = 0}$$

THE LORENZ GAUGE

This can actually be done, thanks to the gauge (coordinate choice) freedom given by the slowly varying coordinate transformations. In fact:

$$\bar{h}_{\mu\nu} \longrightarrow \tilde{\bar{h}}_{\mu\nu} = \bar{h}_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial_\rho \xi^\rho) \quad (1)$$

so that $\partial^\nu \bar{h}_{\mu\nu} \rightarrow \widetilde{(\partial^\nu \bar{h}_{\mu\nu})} = \partial^\nu \bar{h}_{\mu\nu} - \Box \xi_\mu$

Therefore, if we choose the coordinate transformation such that $\Box \xi_\mu = \partial^\nu \bar{h}_{\mu\nu}$ we can transform to the Lorenz gauge.

The equation $\Box \xi_\mu = f_\mu(x)$ always admits solutions

$$\xi_\mu(x) = \int d^4y \, G(x-y) \, f_\mu(x) \qquad \text{with } \Box_x G(x-y) = \delta^4(x-y)$$

(the Green function) and therefore for every $\partial^\nu \bar{h}_{\mu\nu} = f_\mu$ one can always find a $\xi_\mu(x)$ that allows to pass to the Lorenz gauge. In this gauge, it is apparent that Einstein equations take the form of a <u>WAVE EQUATION</u> with source the energy mom. tensor

$$\boxed{\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}}$$

before concluding that this implies the existence of GWs, one should first uniquely fix the coordinates (otherwise said, eliminate all the gauge freedom) because otherwise one is not sure that $h_{\mu\nu}$ contains only physical degrees of freedom. Let us count how many there are up to now:

$$\bar{h}_{\mu\nu} = \bar{h}_{\nu\mu} \quad \Rightarrow \quad 10$$

Lorenz gauge $\quad \partial_\mu \bar{h}^{\mu\nu} = 0 \quad \Rightarrow \quad 10 - 4 = 6$

6 $\underbrace{\text{d.o.f.}}_{\text{independent components}}$ that appear to be radiative. Is this true?

Let us place ourselves <u>OUTSIDE THE SOURCE</u>:

[we can do this since we are in linearized theory over a flat background (for example, we could not do this in the FRW universe, or at least it is not that obvious).]

$$\boxed{\Box \bar{h}_{\mu\nu} = 0}$$

From the transformation

$$\partial^\nu \bar{h}_{\mu\nu} \to \widetilde{(\partial^\nu \bar{h}_{\mu\nu})} = \partial^\nu \bar{h}_{\mu\nu} - \Box \xi_\mu$$

it is clear that we can re-transform chosing the coordinate transformation:

$$x^\mu \to \tilde{x}^\mu = x^\mu + \bar{\xi}^\mu \qquad \text{with} \qquad \Box \bar{\xi}^\mu = 0$$

If we do this, we remain in the Lorentz gauge since $\widetilde{(\partial^\nu \bar{h}_{\mu\nu})} = 0$. Let us then choose $\bar{\xi}^\mu$ with $\Box \bar{\xi}^\mu = 0$. Then, one also has

$$\Box \left(\partial_\mu \bar{\xi}_\nu + \partial_\nu \bar{\xi}_\mu - \eta_{\mu\nu} \partial_\rho \bar{\xi}^\rho \right) = 0$$

Defining $\bar{\xi}_{\mu\nu} = \partial_\mu \bar{\xi}_\nu + \partial_\nu \bar{\xi}_\mu - \eta_{\mu\nu} \partial_\rho \bar{\xi}^\rho$ one has then, from the transformation law of $\bar{h}_{\mu\nu}$ (1),

$$\Box \left(\tilde{\bar{h}}_{\mu\nu} \right) = \Box \left(\bar{h}_{\mu\nu} - \bar{\xi}_{\mu\nu} \right) = 0$$

Therefore, one can choose the coordinate transformation functions $\bar{\xi}_\mu$ such as to impose four conditions on $\bar{h}_{\mu\nu}$ by means of $\bar{\xi}_{\mu\nu}$: four as the components of $\bar{\xi}_\mu$.

1) $\bar{h} = 0$ (which therefore means $h_{\mu\nu} = \bar{h}_{\mu\nu}$)

2) $h^{0i} = 0$ (which from the Lorentz gauge condition now applied on $h_{\mu\nu}$ gives:
$\partial_\nu h^{\mu\nu} = \partial_0 h^{00} + \partial_i h^{0i} = \partial_0 h^{00} = 0$)

From $\partial_0 h^{00} = 0$ we have for free that $h^{00} = 0$, because it is a non-dynamical component that is irrelevant from the point of view of GWs – it is actually a static Newtonian potential.

We are therefore left with $\boxed{6 - 4 = 2}$ independent components of the metric, which represent the two physical degrees of freedom of GWs.

We have defined the

TRANSVERSE TRACELESS GAUGE

$$\boxed{h^i{}_i = 0 \qquad \partial_i h^{ij} = 0 \qquad h^{0\mu} = 0}$$

In which GWs are represented by <u>the two space elements of the symmetric matrix</u> $\boxed{h_{ij}^{TT}}$.

* Choosing the TT gauge is not necessary, but still very convenient: the metric perturbation in this gauge only contains the physical information about the gravitational radiation.

* The TT gauge exhibits the fact that GWs have only two polarisation components (as we will see)

* Note that as we have derived it, the TT gauge cannot be chosen inside the source but only in vacuum. However, the fact that GWs have only two polarisations is true in general. It is in fact possible to extract the two radiative degrees of freedom of GWs also without making the assumption that $T_{\mu\nu}=0$. For this general derivation, see section 2.2 of Flanagan & Hughes gr-qc/0501041

If $T_{\mu\nu} \neq 0$, the metric contains in general

1) gauge degrees of freedom
2) physical d.o.f which are non radiating
3) gravitational waves (the TT part of $h_{\mu\nu}$)

In order to make the physical d.o.f. apparent one splits the metric into irreducible components under rotation and constructs gauge invariant variables which are scalar, vector and tensor (TT) components. Writing down Einstein eqs. for these components, one finds that only the TT (tensor) ones obey a wave equation, while the physical non-radiating d.o.f., which are the scalar and vector ones, obey Poisson-like equations. Therefore, it is possible to demonstrate also when $T_{\mu\nu} \neq 0$ that the radiative d.o.f correspond only to a piece of the metric perturbation that satisfies the TT conditions.

Now let's go back to the case $T_{\mu\nu} = 0$ for simplicity.

$$\Box h_{ij}^{TT}(\underline{x}, t) = 0$$

has <u>plane-wave solutions</u> moving at the speed of light

$$\boxed{h_{ij}^{TT}(\underline{x}, t) = e_{ij}(\underline{k})\, e^{i k_\mu x^\mu}}$$

with $k^\mu = \left(\frac{\omega}{c}, \underline{k}\right)$ and $|\underline{k}| = \frac{\omega}{c}$

$e_{ij}(\underline{k})$ are the <u>polarisation tensors</u>.

If $\underline{k} = |\underline{k}|\,\hat{n}$ the tensor h_{ij}^{TT} has non-zero components only in the plane perpendicular to $\hat{n}$ (transversality of the waves) because of the Lorenz gauge choice:

$$\partial^j h_{ij} = 0 \quad \Rightarrow \quad n^j e_{ij} = 0$$

The polarisation tensors can be rewritten in terms of $(\hat{u}, \hat{v})$ unit vectors living in the plane perpendicular to $\hat{n}$ and perpendicular among each other:

$$e^{+}_{ij} = \hat{u}_i \hat{u}_j - \hat{v}_i \hat{v}_j \qquad e^{\times}_{ij} = \hat{u}_i \hat{v}_j + \hat{u}_j \hat{v}_i$$

choosing a reference frame for which $\hat{n} \parallel z$ one gets (with $a, b = x, y$)

$$e^{+}{}_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{ab} \qquad e^{\times}{}_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{ab}$$

And the wave $h_{ij}^{TT}(z,t)$ becomes then:

$$h_{ij}^{TT}(z,t) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \cos\left[\omega\left(t - \frac{z}{c}\right)\right]$$

the ds^2 of the linearised metric becomes then:

$$ds^2 = (\eta_{\mu\nu} + h_{\mu\nu})\, dx^\mu dx^\nu =$$

$$= -c^2 dt^2 + dz^2 + \left[1 + h_+ \cos[\omega(t - z/c)]\right] dx^2$$

$$+ \left[1 - h_+ \cos\left[\omega\left(t - \frac{z}{c}\right)\right]\right] dy^2 +$$

$$+ 2 h_\times \cos\left[\omega\left(t - \frac{z}{c}\right)\right] dx\, dy$$

One can also find the tensor that projects
on the TT gauge a plane wave propagating in the direction $\hat{n}$, in vacuum and already in the Lorentz gauge:

$$\boxed{h_{ij}^{TT} = \Lambda_{ijkl}\, h_{kl}}$$

with $\quad \Box h_{ij}^{TT} = 0$

$$\Lambda_{ijkl}(\hat{n}) = P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl}$$

with $\quad P_{ij}(\hat{n}) = \delta_{ij} - \hat{n}_i \hat{n}_j$

these are both projectors and transverse:

- $P_{ij} \hat{n}_j = 0 \quad , \quad P_{ij} P_{jk} = P_{ik}$
- $\Lambda_{ijkl} \hat{n}^i = \Lambda_{ijkl} \hat{n}^j = \Lambda_{ijkl} \hat{n}^k = \Lambda_{ijkl} \hat{n}^l = 0$

  $\Lambda_{ijkl} \Lambda_{klmn} = \Lambda_{ijmn}$

and Λ_{ijkl} is also traceless: $\quad \Lambda_{iikl} = \Lambda_{ijkk} = 0$

(while $P_{ii} = 2$)

NOTE : The polarisation states of a classical radiation field can be related to the SPIN of the massless particle that one expects upon quantisation of the theory.

If the polarisation modes are invariant under rotation of an angle Θ, the spin of the particle associated with the radiation field should satisfy

$$S = \frac{2\pi}{\Theta}$$

(Misner, Thorne, Wheeler chapter 35.6)

Example:
EM radiation has two linear polarisation states, i.e. two independent polarisation vectors that are invariant under rotation of 2π $\Rightarrow$ the photon has spin $S = 1$.

We have seen that GWs have 2 independent polarisation states represented by e_{ij}^{+} and $e_{ij}^{\times}$.

A GW propagating in the z direction changes upon rotation around the z-axis of an angle Θ as:

$$h_{ij}^{TT\,\prime} = R_{ik} R_{jl} h_{kl}^{TT}$$

with $R = \begin{pmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ rotation matrix

$$\begin{cases} h'_+ = h_+ \cos 2\theta - h_\times \sin 2\theta \\ h'_\times = h_+ \sin 2\theta + h_\times \cos 2\theta \end{cases}$$

invariant for $\theta = \pi$

Therefore, one expects the graviton to have spin 2:

$$S = \frac{2\pi}{\pi} = 2$$

INTERACTION OF GWs WITH TEST MASSES AND PRINCIPLES OF GW DETECTORS

Within linearized theory the bodies generating the GWs (a binary star for example) are taken to move in flat spacetime: $\eta_{\mu\nu}$. Therefore their dynamics is described with Newtonian gravity, rather than full GR. The emission of GWs does not back-react on the system that produces them, and in fact one has conservation of the source energy momentum tensor $\partial_\mu T^{\mu\nu} = 0$ (from the Lorentz gauge condition and the wave equation) just as in flat spacetime. On the other hand, the response of test masses to GWs are described by the full metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ as we now see.

let us pick as frame the TT frame.

In these coordinates, let us consider a test mass at rest at $\tau=0$: $\left.\frac{dx^i}{d\tau}\right|_{\tau=0} = 0$

the geodesic equation for this mass in the full metric $\eta_{\mu\nu} + h_{\mu\nu}$ and in the TT frame is:

$$\left.\frac{d^2x^i}{d\tau^2}\right|_{\tau=0} = \left[-\Gamma^i_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\right]_{\tau=0} \overset{\text{initially at rest}}{=} \left[-\Gamma^i_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau}\right]_{\tau=0}$$

$$\Gamma^i_{00} = \frac{1}{2}\left(2\partial_0 h^i{}_0 - \partial^i h_{00}\right) \equiv 0$$ since in TT gauge $h_{00}^{TT} = h_{0i}^{TT} = 0$

therefore a <u>mass initially at rest REMAINS AT REST also after the arrival of the GW!</u>

This is an artifact of the choice of coordinates: it does not mean that the passing GW has no physical effect on the test mass, but only that in the TT frame the coordinates are chosen in such a way that they stretch themselves as the GW passes through. It is as if the coordinates were marked by a network of freely falling masses: by definition they do not change as the

GWs passes through because GWs are pure "curvature" and not an external non-gravitational force.

Therefore, to grasp the <u>physical effect</u> of GWs on test masses one has to look at <u>proper distances and proper time</u> : this is a manifestation of the fact that GR is invariant under coordinate exchange, and the physics is independent on the choice of coordinates. Let us consider now two masses at coordinate positions:

$$(t, x_1, 0, 0)$$
$$(t, x_2, 0, 0)$$

the coordinate distance $\xi = x_2 - x_1$ remains constant in the TT gauge under the effect of a passing GW. The proper distance though:

$$ds^2 \underset{\text{in the TT gauge}}{=} -c\,dt^2 + dz^2 + [1 + h_+ \cos[\omega(t - \tfrac{z}{c})]]\,dx^2 +$$
$$+ [1 - h_+ \cos[\omega(t - \tfrac{z}{c})]]\,dy^2 + 2\,h_\times \cos[\omega(t - \tfrac{z}{c})]\,dx\,dy$$

and therefore:

$$S = \sqrt{1 + h_+ \cos \omega t}\,(x_2 - x_1) \underset{|h_+| \ll 1}{\simeq} \xi\,(1 + \tfrac{1}{2} h_+ \cos \omega t)$$

The proper distance between the masses changes periodically as the GW passes through!

generalising to arbitrary directions: $\underline{x}_2 - \underline{x}_1 = \underline{L}$

$$s^2 = L^2 + h_{ij}^{TT} L_i L_j$$

$$s \simeq L\left(1 + \frac{1}{2} h_{ij}^{TT} \frac{L_i L_j}{L}\right) \qquad \text{at first order in } h_{ij}^{TT}$$

$$\ddot{s} \simeq \frac{1}{2} \ddot{h}_{ij}^{TT} \frac{L_i L_j}{L}$$

therefore in the direction i $\quad (s_i = s n_i$ and $L_i = L n_i)$:

$$\ddot{s}_i \simeq \frac{1}{2} \ddot{h}_{ij}^{TT} L_j \simeq \frac{1}{2} \ddot{h}_{ij}^{TT} s_j$$

since $L_j \simeq s_j + O(h)$

$$\boxed{\ddot{s}_i \simeq \frac{1}{2} \ddot{h}_{ij}^{TT} s_j} \Longrightarrow$$

equation expressing what happens to proper distances in the TT frame as a GW passes through: even if the masses are at rest initially, their proper distance changes as the GW passes.

One can therefore <u>MONITOR THE ARRIVAL OF GWS</u> by measuring the changes in the proper distance between two suspended masses: the principle of INTERFEROMETRIC DETECTION, as we will see.

However, what is the proper distance in the reference frame of a detector on Earth?

On Earth we are accelerated observers: we experience the gravitational acceleration and the apparent forces due to Earth's rotation. These forces enter at first order in x in the metric, while GWs (curvature) enter only at second order in x.

The reason for this is the equivalence principle: in GR it is always possible to eliminate the gravitational field locally and choose a frame, THE LOCAL INERTIAL FRAME, such that space-time is locally flat and the metric is:

$$g_{\mu\nu} = g_{\mu\nu}(P) + \frac{1}{2} g_{\mu\nu,\alpha\beta} (x^\alpha - x^\alpha_P)(x^\beta - x^\beta_P)$$

metric Taylor expanded around P

At point P:

$$\begin{cases} g_{\mu\nu}(P) = \eta_{\mu\nu} \\ g_{\mu\nu,\alpha}(P) = 0 \end{cases}$$

$\Rightarrow$ no gravitational field

but CURVATURE EFFECTS cannot be eliminated by the choice of a reference frame, not even the local inertial one, and in fact they appear at second order in x:

$$g_{\mu\nu,\alpha\beta}\,(x^\alpha - x^\alpha_P)(x^\beta - x^\beta_P)$$

If we were able to put our detector as to "coincide" with a local inertial frame on Earth, it would be sensitive to GWs. Is this possible? One would have to eliminate the effects of acceleration and rotation. This is done by compensating these forces with suspension mechanisms which are very refined and leave the masses free to move in the (x, y) directions. The other disturbances such as seismic noise, Earth rotation etc are eliminated by choosing an adapted frequency window:

$0.1\text{ Hz} < f < 10^3\text{ Hz}$ $\Rightarrow$ in this window, the detector is isolated from noises due to the time-varying Earth gravitational field. Moreover, there are GW sources emitting in this window!

For terrestrial detectors

It is therefore possible, through these (very difficult to implement) technical methods, to construct a detector on Earth. The detector can be described in the effective metric which is the one of a freely-falling observer (local inertial frame) in which one retains only the part proportional to the Riemann tensor (the curvature, so the GWs)

$$\left.\begin{aligned} ds^2 \simeq & -c^2 dt^2 \left[1 + R_{0i0j}\, x^i x^j\right] \\ & + 2\,dt\,dx \left[-\tfrac{2}{3} R_{0ijk}\, x^j x^k\right] \\ & + dx^i dx^j \left[\delta_{ij} - \tfrac{1}{3} R_{ikjl}\, x^k x^l\right] \end{aligned}\right\} \text{feat up to second order in } x$$

In this metric, proper time and proper distances are the SAME as coordinate time and coordinate distances <u>at first order in Riemann, therefore in h_{ij}</u>. We can therefore analyse the displacement caused by the passing wave on test masses using the equation:

$$\boxed{\ddot{\xi}_a(t) = \frac{1}{2}\,\ddot{h}_{ab}^{TT}\,\xi_b(t)} \qquad (2)$$

coordinate distance = proper distance

The effect of a GW on a circle of test masses :

We consider a GW propagating in the z direction: since $h_{ij}^{TT} = 0$ for $i = 3$ and $j = 3$, particles are not displaced in the z direction: GWs displace masses <u>transversally</u> with respect to their propagation direction.

At $z=0$ for + polarization with $h_{ab}^{TT}(t=0) = 0$ we can write:

$$h_{ab}^{TT}(z=0,t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{ab} h_+ \sin(\omega t)$$

the coordinate distance between masses is

$$\xi_a(z=0,t) = (x_0 + \delta x(t),\ y_0 + \delta y(t),\ 0)$$

Applying eq (2) one has :

$$\begin{cases} \ddot{\delta x}(t) = -\dfrac{h_+}{2}(x_0 + \delta x(t))\,\omega^2 \sin\omega t \\ \ddot{\delta y}(t) = \dfrac{h_+}{2}(y_0 + \delta y(t))\,\omega^2 \sin\omega t \end{cases}$$

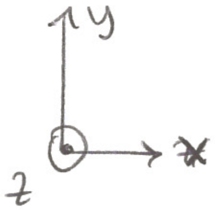
Neglecting the displacements on the r.h.s. since they are of order h, one solves:

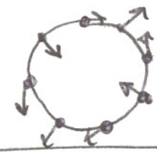
$$\begin{cases} \delta x(t) = \dfrac{h_+}{2} x_0 \sin\omega t \\ \delta y(t) = -\dfrac{h_+}{2} y_0 \sin\omega t \end{cases}$$

For the cross polarisation one finds:

$$\begin{cases} \delta x(t) = \dfrac{h_\times}{2}\, y_0 \sin\omega t \\[2ex] \delta y(t) = \dfrac{h_\times}{2}\, x_0 \sin\omega t \end{cases}$$

therefore on a ring of test masses:

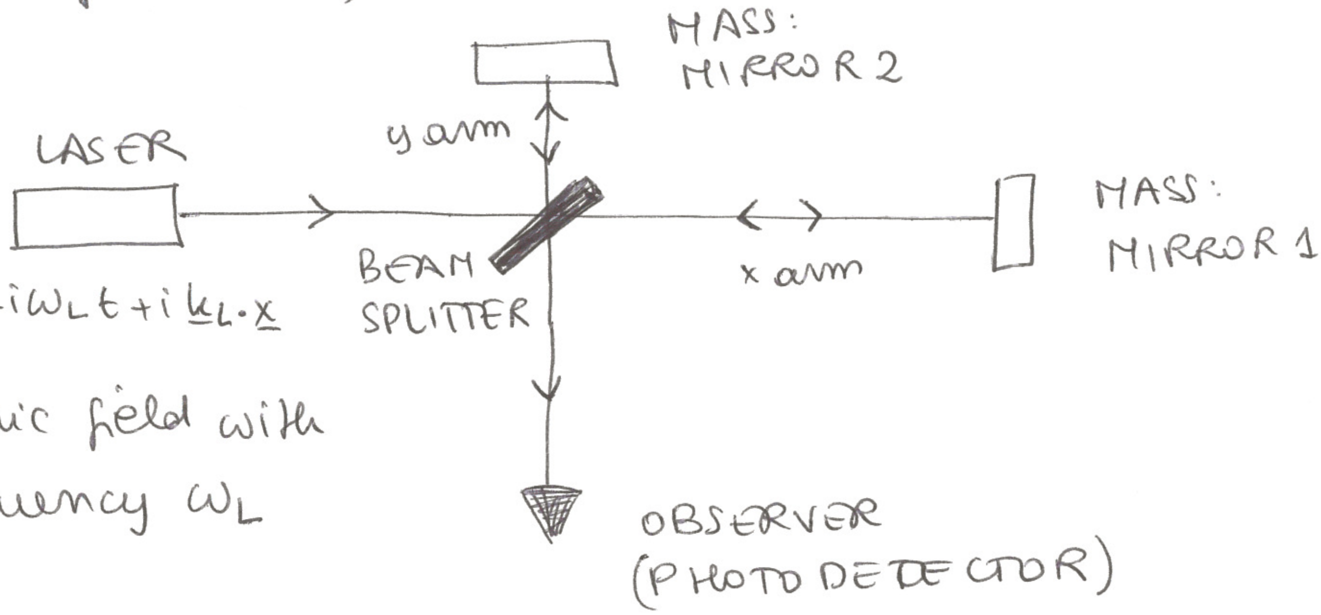


ωt	h_+	$h_\times$
0		
$\frac{\pi}{2}$		
π		
$\frac{3\pi}{2}$		

In an interferometer there are two test masses, as we are now going to discuss.

PRINCIPLES OF GW DETECTOR

(interferometer)



$E_0 \, e^{-i\omega_L t + i \underline{k}_L \cdot \underline{x}}$

Electric field with frequency ω_L

The instrument measures changes in the travel time of the laser light in the two arms: at the <u>beam splitter position</u> ($\underline{x} = 0$) and at <u>observation time t</u> we have the superposition of a beam that entered the beam splitter at a time:

$$t_0^x = t - 2\frac{L_x}{c} \qquad \text{(beam that went through the x arm)}$$

$$t_0^y = t - 2\frac{L_y}{c} \qquad \text{(beam that went through the y arm)}$$

Since the phase* of the laser does not change during the propagation, the electric fields that combine at the beam splitter are:

* the value of the exponent in the electric field

$$\begin{cases} E_1 = -\frac{1}{2} E_0 \, e^{-i\omega_L \left(t - 2\frac{L_x}{c}\right)} \rightarrow \text{initial phase, when the photon entered the x-arm} \\ E_2 = \underbrace{\frac{1}{2}}_{\text{coefficients due to reflection}} E_0 \, e^{-i\omega_L \left(t - 2\frac{L_y}{c}\right)} \rightarrow \text{when it entered the y-arm} \end{cases}$$

giving an output:

$$E_1 + E_2 = -i E_0 \, e^{-i\omega_L t + i\frac{\omega_L}{c}(L_x + L_y)} \sin\left[\frac{\omega_L}{c}(L_y - L_x)\right]$$

$$|E_1 + E_2|^2 = E_0^2 \sin^2\left[\frac{\omega_L}{c}(L_y - L_x)\right] \qquad (3)$$

therefore the power at the photodetector varies as the difference in lenght between the arms varies. This is a <u>physical effect</u> that must be the same regardless of the reference frame: in the <u>detector frame</u> it is clear, since the GWs passing displace the masses (the mirrors at the end of the arms) from their original position and change therefore the lenght of the arms causing the varying power output. If we choose to describe the detector in the <u>TT gauge</u>, we must have the same effect: in this case, the mirrors (test masses) are in free-fall and their coordinate distance remains the same; the effect of the GWs manifests itself by affecting THE PROPAGATION OF THE LIGHT between the two test masses.

Let us analyse the situation in the TT gauge, (26)
supposing a wave that has only + polarisation
and comes from the z direction (detector in x, y):

$$h_+(t) = h_0 \cos(\omega t)$$

(where ω is ω_{GW}, not to be confused with ω_L of the laser)

$$ds^2 = -c^2 dt^2 + [1 + h_+(t)] dx^2 + [1 - h_+] dy^2 + dz^2$$

The photons of the laser travel on null geodesics, and therefore for the light in the x-arm one gets:

$$ds^2 = 0 \quad \Rightarrow \quad dx \simeq \pm\, c\,dt \left[1 - \frac{1}{2} h_+(t)\right]$$

(Taylor expansion in h_+)

$$\underbrace{\int_0^{L_x} dx - \int_{L_x}^{0} dx}_{\text{full photon path}} = \underbrace{\int_{t_0^x}^{t_1} c\,dt' \left[1 - \frac{h_+}{2}\right]}_{\text{going}} - \underbrace{\int_{t_1}^{t} -c\,dt' \left[1 - \frac{h_+}{2}\right]}_{\text{coming back}}$$

$$2L_x = \int_{t_0^x}^{t} c\,dt' \left[1 - \frac{h_+}{2}\right] = c\,(t - t_0^x) - \frac{c}{2} \int_{t_0^x}^{t} dt'\, h_+(t')$$

The total time interval that the photon spends in the x-arm (which has fixed lenght in TT frame) is therefore :

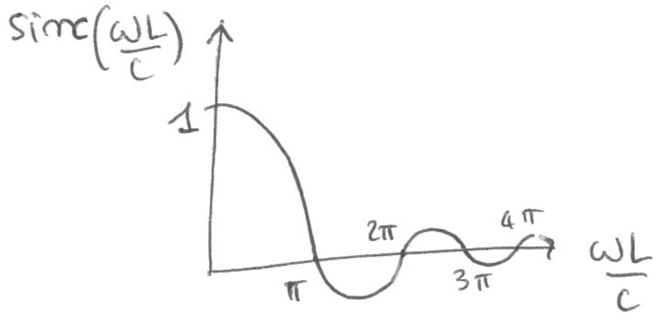
(inverting the above and expanding for h_+)

$$t - t_0^x \simeq \frac{2L_x}{c} + \frac{1}{2}\int_{t_0^x}^{t_0^x + 2L_x/c} dt'\, h_+(t')$$

$\Rightarrow$ here one neglects the effect of GWs which would give a second order term $t = t_0^x + 2\frac{L_x}{c}$

$$\simeq \frac{2L_x}{c} + \frac{L_x}{c}\,\frac{\sin\left(\frac{\omega L_x}{c}\right)}{\frac{\omega L_x}{c}}\, h_+\left(t_0^x + \frac{L_x}{c}\right)$$

So there is a correction in the time interval from the value $2\frac{L_x}{c}$ due to the presence of the GW. The function that determines the amplitude of the correction (proportional to h_+ of course) is such that:



- if $\frac{\omega L_x}{c} \ll 1$ the time shift is just h_+
- if $\frac{\omega L_x}{c} \gg 1$ the effect cancels out since the GW does many oscillations

One can now calculate the output in the detector accounting for the fact that the phase does not change. For this, one expresses from the above t_0^x and t_0^y as functions of the parameters:

$$t_0^x \simeq t - \frac{2L_x}{c} - \frac{L_x}{c} \operatorname{sinc}\left(\frac{\omega L_x}{c}\right) h_+\left(t - \frac{L_x}{c}\right)$$

again, neglect GW effect there

$$t_0^y \simeq t - \frac{2L_y}{c} - \frac{L_y}{c} \operatorname{sinc}\left(\frac{\omega L_y}{c}\right) h_+\left(t - \frac{L_y}{c}\right)$$

and the total electric field at the beam splitter at time t becomes: (we set $L_x \simeq L_y$ in the part linear in h_+)

$$E_1 + E_2 = -i E_0 e^{-i\omega_L\left(t - \frac{2L}{c}\right)} \sin\left[\phi_0 + \Delta\phi_x(t)\right]$$

$\phi_0 = \frac{\omega_L}{c}(L_x - L_y)$ phase to adjust, not relevant

effect of GWs, dephasing

with

$$\boxed{\Delta\phi_x(t) = \frac{\omega_L L}{c} \operatorname{sinc}\left(\frac{\omega L}{c}\right) h_+\left(t - \frac{L}{c}\right)}$$

So that the difference in lenght is about $\frac{\Delta L}{L}(t) \propto h_+(t)$ as one would have expected. (Eq. (3))

In order to MAXIMISE the effect, one must choose

$$\mathrm{Max}\left[\sin\left(\frac{\omega L}{c}\right)\frac{\omega_L}{\omega}\right] \Rightarrow \boxed{L = \frac{\pi}{2}\frac{c}{\omega}}$$

To be sensitive to a frequency of about 100 Hz, which is rich of sources and within the interval in which one can get rid of seismic noises, one therefore should choose:

$$\boxed{L \simeq 750\ \text{km} \left(\frac{100\ \text{Hz}}{f}\right)}$$

with $f = \omega c$. This is clearly not possible on Earth: the Earth-based interferometers have arms of 4 km (advanced LIGO) and 3 km (advanced VIRGO). They use FABRY-PEROT CAVITIES that effectively fold the 750 km lenght into 3/4 km by reflecting the laser many times back and forth. The effective lenght becomes then $\frac{2F}{\pi}$ times larger, where F is the "finesse" of the cavity.

$$L_{eff} \simeq \frac{\pi}{2F}\ 750\ \text{km} \simeq 4\ \text{km} \quad \text{for} \quad F \simeq \mathcal{O}(10^2)$$

The dephasing one needs to measure is:

$$2\Delta\phi_x \simeq \frac{2\omega_L}{c} \frac{2F}{\pi} L_{eff}\, h_+ \simeq 6.4 \cdot 10^{13}\, h_+ \simeq \boxed{10^{-8}\ \text{rad}}$$

total

here we assume $\frac{\omega L}{c} \ll 1$ which is always true for Earth based interferometers

$h_+ \simeq 10^{-21}$ is a typical value, as we will see.

$$f \simeq 10^2 - 10^3\ \text{Hz} \Rightarrow \bar{\lambda} = \frac{\lambda}{2\pi} \simeq 500 - 50\ \text{km} \gg L \simeq 4\ \text{km}$$

THE ENERGY MOMENTUM TENSOR OF GWs

From the fact that GWs displace test masses it is clear that they carry energy and momentum. What is the expression for the en-mom. TENSOR of GWs? According to GR, every form of energy contributes to the curvature of space-time.

Are GWs a source of space-time curvature? The answer to this question will allow us to find an expression for the GW energy-mom. tensor.

However, to answer this question one must GO BEYOND LINEARISED THEORY on flat spacetime

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

1. this is defined as GW because it is the part of the metric that satisfies a wave equation in the Lorenz gauge

2. but keeping Minkowski as the background one excludes since the beginning that GWs can create curvature on the background spacetime

$\Rightarrow$ go beyond linearised theory to define the energy mom. tensor of GWs.

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1$$

$\uparrow$
curved
dynamical
background
metric

However in this setting it is non-trivial to decide what is the background and what is the fluctuation. To distinguish them, one needs to do a further assumption: there must be a neat separation of SCALES / FREQUENCIES

$\bar{g}_{\mu\nu}$ typical scale of spatial variations $\qquad + \qquad$ $h_{\mu\nu}$ small amplitude perturbations with reduced wavelenght

$$\boxed{L_B \gg \lambda\!\!\!^{-}} = \frac{\lambda}{2\pi}$$

if this is satisfied then $h_{\mu\nu}$ has the meaning of SMALL RIPPLES on a smooth background
(ex: waves on the sea
skin of an orange)

in terms of frequencies, one would say that the typical time scale of variation of $\bar{g}_{\mu\nu}$ has characteristic frequency f_B while for the waves $h_{\mu\nu}$ the characteristic frequency is $f = \frac{c}{\lambda}$; if it is satisfied that $\boxed{f \gg f_B}$ then $h_{\mu\nu}$ can be distinguished from the background as a small, rapidly varying perturbation on a static or slowly varying background.

* Note that $f = c/\lambda$ but it needn't be $f_B = c/L_B$

* Note that $f_B \ll f$ is true for Earth based detectors because $f \sim 10^2 - 10^3$ Hz is much higher than the typical frequency of variation of the Earth gravitational field - earthquakes. On the other hand, $L_B \gg \lambda$ is not verified since $\lambda \simeq 500 - 50$ km and the Earth gravitational field is not smooth over this scale - mountains etc. However, the condition $f_B \ll f$ alone is enough to distinguish the GWs from the rest and perform the measurement.

Now let us assume that either $f_B \ll f$ or $L_B \gg \lambda$ is verified and <u>investigate how the perturbation $h_{\mu\nu}$ affects the background</u> :

To answer this question, we start with Einstein eqs and expand them

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)$$

and there are two small parameters for the expansion:

1) $|h_{\mu\nu}| \ll 1$

2) $\frac{\lambda\!\!\!^{-}}{L_B} \ll 1$ $\left(\text{or } \frac{f_B}{f} \ll 1\right)$

we start by expanding at SECOND ORDER IN $h_{\mu\nu}$: we need to go beyond linearised theory, but second order is enough to define the energy momentum tensor.

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu}$$

$\bar{R}_{\mu\nu}$	$R^{(1)}_{\mu\nu}$	$R^{(2)}_{\mu\nu}$
part constructed only with $\bar{g}_{\mu\nu}$: it varies on the same lenght scale as the background L_B (frequency f_B)	part which is linear in $h_{\mu\nu}$: it varies on the same lenght scale as $h_{\mu\nu}$ $\lambda\!\!\!^{-}$ (frequency f)	term quadratic in $h_{\mu\nu}$: since it is quadratic it contains BOTH long and short variation scales so it can contribute to the background curvature (both low and high frequencies)

if we want to isolate the part of Einstein eqs that only contains long lenght-scales (low frequencies) in order to investigate the effect of GWs on the background we have to include $R_{\mu\nu}^{(2)}$:
Einstein eqs for the background metric become

$$\bar{R}_{\mu\nu} = \left[- R_{\mu\nu}^{(2)} \right]^{low} + \frac{8\pi G}{c^4} \left[T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right]^{low} \quad (4)$$

$\bar{R}_{\mu\nu}$: background curvature

$\left[- R_{\mu\nu}^{(2)} \right]^{low}$: source of bck curvature from the "low frequency" "long wavelenght" part coming from the second order in $h_{\mu\nu}$: it produces an effect on $\bar{R}_{\mu\nu}$ which is of the order:

$$R_{\mu\nu}^{(2)} \sim (\partial h)^2 + h\,\partial^2 h \sim O\left(\frac{h}{\lambda}\right)^2$$

$\frac{8\pi G}{c^4} \left[T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right]^{low}$: part of the external source $T_{\mu\nu}$ of bck curvature that varies on long wavelenght / small frequencies : it causes the bck curvature that varies on scale L_B / on frequency f_B that we accounted for initially.
It has nothing to do with GWs.

the effective method to perform the distinction among GWs and the background based on their typical lenght scales / frequencies is to PERFORM AN AVERAGE:

•) select a scale $\bar{\ell}$ such that $\bar{\lambda} \ll \bar{\ell} \ll L_B$ and average quantities on a volume $\bar{\ell}^3$: all the short-wavelenghts modes will average to zero.

•) select a frequency $\bar{f}$ such that $f_B \ll \bar{f} \ll f$ and average quantities over a time-scale $\bar{t} = 1/\bar{f}$, i.e. over several periods of the GWs. All high frequencies modes will average to zero.

For what concerns the matter part, one has then

$$\left[T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right]^{low} = \left\langle T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right\rangle := \bar{T}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{T} \tag{5}$$

In order to get the energy-momentum tensor of GWs, we define the quantity:

$$\boxed{t_{\mu\nu} := -\frac{c^4}{8\pi G} \left\langle R^{(2)}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} R^{(2)} \right\rangle}$$

(with $R^{(2)} = \bar{g}_{\mu\nu} R^{(2)\mu\nu}$)

From this definition, one gets by inverting it (note that $\bar{g}_{\mu\nu}$ exits and enters the average without problems):

$$\langle R^{(2)}_{\mu\nu} \rangle = -\frac{8\pi G}{c^4}\left[t_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\, t \right] \qquad (6)$$

Inserting now eqs (5) and (6) into eq (4) one gets

$$\bar{R}_{\mu\nu} = \frac{8\pi G}{c^4}\left[t_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\, t \right] + \frac{8\pi G}{c^4}\left[\bar{T}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\, \bar{T} \right]$$

and therefore, rearranging:

$$\boxed{\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} = \frac{8\pi G}{c^4}\left[\bar{T}_{\mu\nu} + t_{\mu\nu} \right]}$$

the dynamics of the background metric is determined by

the long scales/ low frequency part of the en. mom. tensor of matter $\bar{T}_{\mu\nu}$ $\boxed{\text{plus}}$ the contribution of GWs represented by the energy mom. tensor $t_{\mu\nu}$ that depends only on $h_{\mu\nu}$ and is QUADRATIC in $h_{\mu\nu}$.

It is non-trivial to evaluate $t_{\mu\nu}$. For a complete derivation, see Maggiore's book or Flanagan & Hughes already cited or Misner, Thorne & Wheeler. Here we only give the final result:

$$\boxed{t_{\mu\nu} = \frac{c^4}{32\pi G} \langle \partial_\mu h_{\sigma\beta} \partial_\nu h^{\sigma\beta} \rangle}$$

This expression for the energy mom. tensor of GWs is free from gauge modes (all the gauge freedom has been eliminated), it is therefore a physical quantity that can only be defined INSIDE AN AVERAGE, either over space, or over time or over both as we have discussed.

The GRAVITATIONAL WAVE ENERGY DENSITY is therefore (written in the TT gauge as usual):

$$\boxed{\rho_{GW} = t^{00} = \frac{c^2}{32\pi G} \langle \dot{h}_{ij}^{TT} \dot{h}_{ij}^{TT} \rangle}$$

The energy flux per unit time and unit surphace of GWs is given by:

$$\frac{dE}{dt\,dA} = \frac{c^3}{32\pi G} \langle \dot{h}_{ij}^{TT} \dot{h}_{ij}^{TT} \rangle \quad (7)$$

(for a derivation see Maggiore)

GENERATION OF GWs IN LINEARISED THEORY

The generation of GWs keeping the expansion around flat space-time (linearized theory) implies that the source must be described by Newtonian theory (otherwise the space-time around the source would be deformed by its gravitational field and deviate from Minkowski). Therefore, <u>the gravitational field generated by the source is assumed to be weak</u>.

For a self gravitating system, weak gravitational field implies <u>small velocities</u>:

two body system held together by the gravitational force :

$$m = m_1 + m_2$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad \text{reduced mass}$$

$$E_{kin} = -\frac{1}{2} U$$

$$\frac{1}{2} \mu v^2 = \frac{1}{2} \frac{G \mu m}{r}$$

$$\left(\begin{array}{c} \text{Schwarzschild radius} \\ R_s = 2 \frac{Gm}{c^2} \end{array} \right)$$

$$\Rightarrow \left(\frac{v}{c} \right)^2 = \frac{R_s}{2r}$$

weak gravitational field $\qquad$ small velocity

$$r \gg R_s \quad \Rightarrow \quad v \ll c$$

(otherwise: Post-Newtonian formalism, not treated here) when solving for GW emission, we will then perform an expansion in the small parameter v/c. Let us first solve the wave equation with a generic source $T_{\mu\nu}(\underline{x},t)$:

(in the Lorentz gauge)

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

the solution is given by the retarded Green function

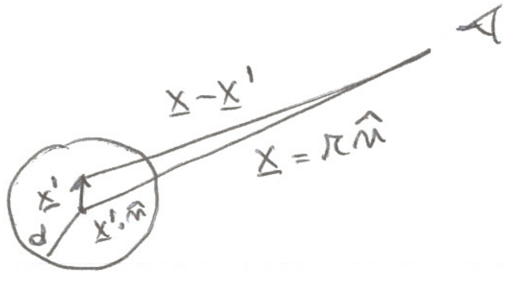
$$G(x-x') = -\frac{1}{4\pi|\underline{x}-\underline{x}'|}\,\delta(x^0_{ret}-x'^0)$$

($x'^0 = ct'$, $x^0_{ret} = c\,t_{ret}$, $t_{ret} = t - \frac{|\underline{x}-\underline{x}'|}{c}$). If we place ourselves OUTSIDE THE SOURCE, we can project on the TT gauge. The full solution becomes then: ($\underline{x} = r\hat{n}$)

(projection on the TT gauge)

$$h^{TT}_{ij}(t,\underline{x}) = \frac{4G}{c^4}\Lambda_{ij\,kl}(\hat{n})\int d^3x'\,\frac{T_{kl}\left(t-\frac{|\underline{x}-\underline{x}'|}{c},\underline{x}'\right)}{|\underline{x}-\underline{x}'|}$$

we now further simplify this formula by noting that the observer can be placed FAR AWAY from the source:



$$|\underline{x}-\underline{x}'| = r - \underline{x}'\cdot\hat{n} + \left(\frac{d^2}{r}\right)$$

neglecting terms of order $\frac{d}{r^2}$ in the amplitude and of order $\left(\frac{d}{r}\right)^2$ in the integral, one gets:

$$h^{TT}_{ij}(\underline{x},t) = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl}(\hat{n})\int d^3x' \, T_{kl}\left(t-\frac{r}{c}+\frac{\underline{x}'\cdot\hat{n}}{c}, \underline{x}'\right)$$

We now perform the <u>low-velocity expansion</u>:

let us assume that we can expand in the small parameter (time) $\frac{\underline{x}'\cdot\hat{n}}{c}$:

$$T_{kl}\left(t-\frac{r}{c}+\frac{\underline{x}'\cdot\hat{n}}{c}, \underline{x}'\right) \simeq T_{kl}\left(t-\frac{r}{c}, \underline{x}'\right) + \frac{x'_i n_i}{c}\partial_0 T_{kl}\Big|_{t-\frac{r}{c}} + \frac{(x'_i n_i)(x'_j n_j)}{2c^2}\partial_0^2 T_{kl}\Big|_{t-\frac{r}{c}} + \dots \quad (8)$$

calling t_s the typical time of variation of the source, one gets

$$\frac{x'_i n_i}{c}\partial_0 T_{kl} \sim \frac{x'_i n_i}{c\, t_s} T_{kl} \leq \frac{d}{c t_s} T_{kl} \simeq \frac{v}{c} T_{kl} \ll T_{kl}$$

so we see that the above expansion is justified by the low-velocity of the source. Neglecting higher order terms, we can simply set

$$h_{ij}^{TT}(\underline{x},t) \simeq \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl}(\hat{n})\int d^3x' \, T_{kl}(t_{ret}, \underline{x}') \quad (9)$$

to put this equation in a more familiar form, let us define the second momentum of the energy density of the source $\rho = T^{00}$:

$$\left(\begin{array}{ll} M = \frac{1}{c}\int d^3x\, T^{00} & \rightarrow \text{first momentum, mass} \\ M_i = \frac{1}{c^2}\int d^3x\, T^{00} x_i & \rightarrow \text{second momentum} \end{array} \right)$$

$$\boxed{M_{ij} = \frac{1}{c^2}\int d^3x\, T^{00}\, x_i\, x_j}$$ second momentum

this can be related to the integral in (8) by using the CONSERVATION OF ENERGY AND MOMENTUM OF THE SOURCE, $\boxed{\partial_\mu T^{\mu\nu} = 0}$

(as we have seen, in linearised theory $\partial_\mu T^{\mu\nu} = 0$ is satisfied, implying that the source is not loosing energy and momentum via GW emission: back-reaction is neglected in linearised theory. This is of course an approximation, and we will see later how to deal with this in the context of inspiralling binaries)

$$\ddot{M}_{ij} = \left(\frac{1}{c}\int d^3x\, \partial_0 T^{00}\, x_i x_j\right)^{\cdot} \overset{\partial_\mu T^{\mu 0} = 0}{=} \left(-\int \frac{d^3x}{c}\, \partial_k T^{0k} x_i x_j\right)^{\cdot}$$

$$= \left(-\int \frac{d^3x}{c}\, \partial_k (T^{0k} x_i x_j) + \int \frac{d^3x}{c}\, T^{0k} \delta_{ki} x_j + \int \frac{d^3x}{c}\, T^{0k} \delta_{kj} x_i\right)^{\cdot}$$

the first term is a total derivative and therefore it does not contribute. One has then:

$$\ddot{M}_{ij} = \int d^3x\ \partial_0 T^{0i} x_j + \int d^3x\ \partial_0 T^{0j} x_i \overset{\partial_\mu T^{\mu i}=0}{=}$$

$$= -\int d^3x\ \partial_k T^{ki} x_j - \int d^3x\ \partial_k T^{kj} x_i = \text{again, total derivatives do not contribute}$$

$$= 2\int d^3x\ T^{ij}$$

therefore, the integral in (9) can be expressed as the double time derivative of the second mass momentum. We have arrived at an important conceptual point: A STATIC SOURCE CANNOT RADIATE GWs.

To get finally to the quadrupole formula, let us note that the tensor M_{ij} can be decomposed into a pure trace and a traceless part, and the Λ_{ijkl} projector will only act on the traceless part, putting the trace to zero: therefore

$$\Lambda_{ijkl}(\hat{n})\,\ddot{M}_{kl} = \Lambda_{ijkl}(\hat{n})\left(M_{kl} - \frac{1}{3}\delta^{kl} M_{pp}\right)^{\cdot\cdot}$$

$$= \Lambda_{ijkl}(\hat{n})\,\ddot{Q}_{kl}$$

where Q_{kl} is the quadrupole moment of the mass density

$$Q_{kl} = \int d^3x\ \rho(\underline{x},t)\left(x_k x_l - \frac{1}{3} r^2 \delta_{kl}\right)$$

and therefore:

$$\boxed{\left[h_{ij}^{TT}(t,\underline{x})\right]_{QUAD} = \frac{1}{r}\,\frac{2G}{c^4}\,\Lambda_{ij\,kl}(\hat{n})\,\ddot{Q}_{kl}\left(t-\frac{r}{c}\right)} \quad (10)$$

GWs are radiated only by non-static mass distributions that possess at least a quadrupole moment. Monopole and dipole radiation is absent for GWs: the reason is that a monopole term would depend on M the mass, and a dipole term would depend on the momentum P^i.

$\left(M = \int d^3x\, \frac{T^{00}}{c^2} \right.$, $\left. P^i = \int d^3x\, \frac{T^{0i}}{c} \right)$ However, these two quantities are conserved, $\dot{M} = \dot{P}^i = 0$. Since a static distribution of mass does not radiate, these contributions must vanish. The first order in the multipole expansion based on the small parameter $\frac{v}{c} \ll 1$ that can emit GWs is given by the quadrupole.

If we had not stopped at the lowest order in (7), we would have found also the octupole and all other multipoles of the v/c expansion.

NOTE : actually, mass and momentum conservation is strictly valid only in linearised theory, within which one assumes that $\partial_\mu T^{\mu\nu} = 0$ and the GW emission does not back-react on the source. In reality, a radiating system loses mass and momentum. However, even fully accounting for back-reaction and non-linearities, still one would find that the monopole and the dipole do not radiate: this characteristic of GW emission goes beyond linearised theory, it is much more general and is in fact a manifestation of the intrinsic nature of GWs: connected to the fact that they only have two physical degrees of freedom represented by the two polarisation states of the TT gauge, and connected to the fact that the graviton has spin 2.

GW EMISSION FROM A BINARY SYSTEM

Let us start by rewriting eq (10) in a form which is more useful for calculations. From the definition of the Λ_{ijkl} projector on page 13, one gets

$$\Lambda_{ijkl}\ddot{Q}_{kl} = (P\ddot{M}P)_{ij} - \frac{1}{2} P_{ij}\, tr(P\ddot{M})$$

→ can use M since trace does not contribute

for a wave propagating in the $\hat{z}$ direction:

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad tr(P\ddot{M}) = \ddot{M}_{11} + \ddot{M}_{22}$$

$$P\ddot{M}P = \begin{pmatrix} \ddot{M}_{11} & \ddot{M}_{12} & 0 \\ \ddot{M}_{12} & \ddot{M}_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and therefore, the polarisation amplitudes of a wave propagating in the $\hat{z}$ direction become:

$$\begin{cases} h_+(t,\hat{z}) = \dfrac{1}{r}\dfrac{G}{c^4}\left(\ddot{M}_{11} - \ddot{M}_{22}\right)\left(t - \dfrac{r}{c}\right) \\[2ex] h_\times(t,\hat{z}) = \dfrac{1}{r}\dfrac{G}{c^4}\,\ddot{M}_{21}\left(t - \dfrac{r}{c}\right) \end{cases} \qquad (11)$$

From these expressions, we can calculate the GW signal emitted by a generic distribution of mass. Let us consider then an <u>ISOLATED SYSTEM composed of two point masses m_1 and m_2 moving on a CIRCULAR ORBIT determined</u>

ONLY BY THE MUTUAL INTERACTION OF THE MASSES. It is important that the system is isolated because in this case the energy momentum tensor is conserved and can be used on the right hand side of the GW equation in linearised theory. The second mass moment for this system becomes:

$$T^{\mu\nu} = m_1 \frac{dx_1^\mu}{dt} \frac{dx_1^\nu}{dt} \delta^3(\underline{x} - \underline{x}_1(t)) + m_2 \frac{dx_2^\mu}{dt} \frac{dx_2^\nu}{dt} \delta^3(\underline{x} - \underline{x}_2(t))$$

($\underline{x}_{1,2}(t)$ are the trajectories of the particles)

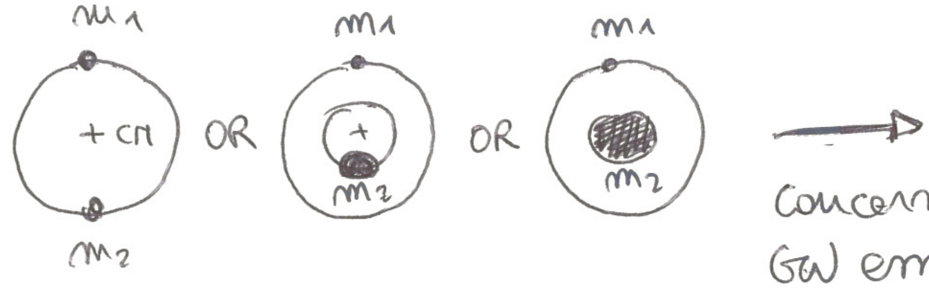
$$M_{ij}(t) = \frac{1}{c^2} \int d^3x \, T^{00} x_i x_j = m_1 x_{1i} x_{1j} + m_2 x_{2i} x_{2j}$$

By choosing the centre of mass frame with $\underline{x}_{CM}(t) = 0$ and mutual particle trajectory $\underline{x}_0(t) = \underline{x}_1(t) - \underline{x}_2(t)$ it is easy to show that

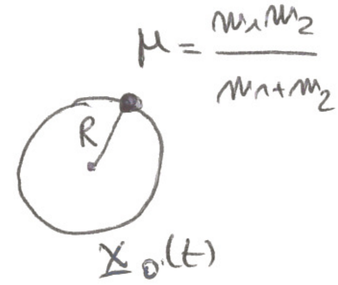
$$M_{ij} = \mu \, \underline{x}_{0i}(t) \, \underline{x}_{0j}(t)$$

which is the second mass moment of a particle with mass corresponding to the reduced mass μ and trajectory corresponding to the relative trajectory $\underline{x}_0(t)$ of the two initial particles.

Therefore, to evaluate the GW emission from a binary system, one can use the M_{ij} of a single mass in circular orbit



Concerning GW emission, these three systems are the same as



Let us therefore take as relative trajectory in the CM frame

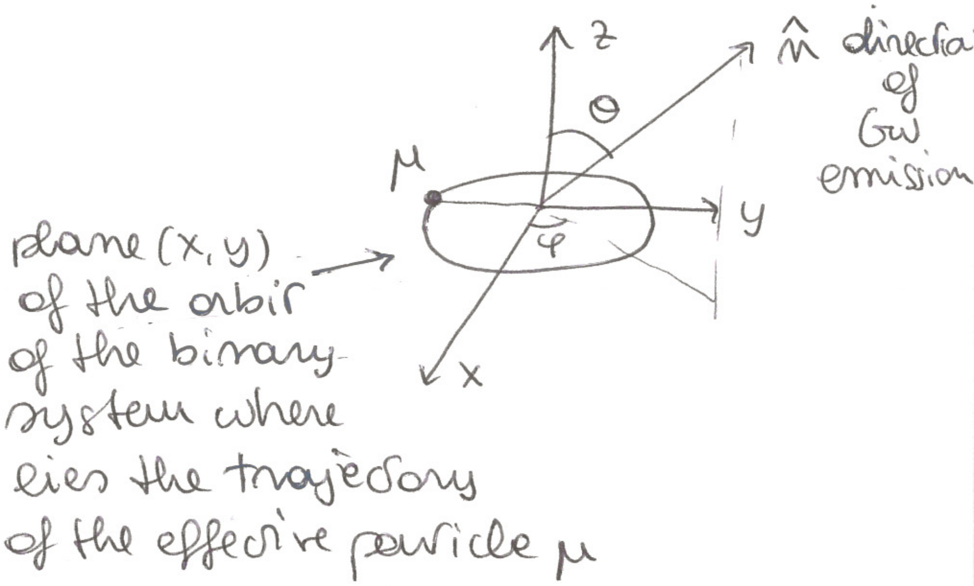
$$\begin{cases} x_0(t) = R\cos\left(\omega_s t + \frac{\pi}{2}\right) \\ y_0(t) = R\sin\left(\omega_s t + \frac{\pi}{2}\right) \\ z_0(t) = 0 \end{cases}$$

the second mass moment is

$$\begin{cases} M_{11} = \mu R^2 \dfrac{1 - \cos(2\omega_s t)}{2} \\ M_{22} = \mu R^2 \dfrac{1 + \cos(2\omega_s t)}{2} \\ M_{12} = \mu R^2 \left(- \dfrac{\sin(2\omega_s t)}{2}\right) \end{cases}$$

by inserting these expressions into (11) one finds the GW signal emitted in the $\hat{z}$ direction. One can generalise (11) as to get the GW signal in an arbitrary direction $\hat{n} = (\theta, \varphi)$ (see for example MAGGIORE chapter 3.3) to finally get, in a general direction from the binary system:

$$\boxed{\begin{aligned} h_+(t,\theta,\varphi) &= \frac{1}{r}\,\frac{4G\mu\omega_s^2R^2}{c^4}\,\frac{1+\cos^2\theta}{2}\cos(2\omega_s t_{ret}+2\varphi) \\ h_\times(t,\theta,\varphi) &= \frac{1}{r}\,\frac{4G\mu\omega_s^2R^2}{c^4}\cos\theta\,\sin(2\omega_s t_{ret}+2\varphi) \end{aligned}} \qquad (12)$$



* from the eqs above, we see that a non-relativistic sources performing harmonic oscillations with frequency ω_s emits monocromatic quadrupole radiation at frequency $2\omega_s$

* the dependence on φ is due to the fact that the system is symmetric under rotations around $\hat{z}$ and a shift in the phase can be reabsorbed by a redefinition of the origin of time

* from the degree of polarisation observed, one can deduce the inclination of the orbit of the emitting system : for example, if $\theta = \frac{\pi}{2}$, we see the orbits edge-on and there is only + polarisation, while if $\theta = 0$ and we see the system face-on the polarisation appears to be circular $h_+ = h_\times$.

* the frequency of the emitted GWs is $2\omega_s$. One has therefore:

$$\bar{\lambda} = \frac{\lambda}{2\pi} = \frac{c}{\omega} = \frac{c}{2\omega_s} \underset{\substack{\uparrow \\ v \sim \omega_s d}}{=} \frac{c}{2}\frac{d}{v} \gg d$$

therefore the wavelenght of GWs emitted by a source of size d is typically much larger than the source size (if the source is non-rel.)

As consequences, <u>we cannot resolve the GW source by GW observation only</u>, it is a different situation than for EM emission.

* The total emitted power can be calcula-ted from eq. (6) of the radiated energy per unit time and surphace inserting eqs (12) and integrating over the angle (note that the r^2 from the surphace cancels with the $1/r^2$ from the amplitude in h_+ and $h_\times$). The final result is:

$$[P]_{QUAD} = \frac{32}{5} \frac{G\mu^2 R^4 \omega_s^6}{c^5} \qquad (13)$$

INSPIRAL OF COMPACT BINARIES

In reality, as the two compact bodies orbit around each other, the GW emission causes the system to loose energy and momentum, inducing a shrinking of the orbit and finally the coalescence. It is actually possible to account for this fact within linearised theory, via a simple expedient as we will see. We can obtain therefore a more realistic description of the GW signal emitted by the binary system during the inspiral phase. On the other hand, the final phases of the inspiral and the merger cannot be described by linearised theory as strong field effects come into play and one needs to use full GR description. Also, in the context of linearised theory one cannot account for effects due to the spin or to the finite size of the bodies. However, what we will do now is good enough to have an idea of the signal detected by the LIGO interferometers.

From the Newtonian dynamics in CM frame one gets:

$$v = \omega_s R \qquad a = \frac{v^2}{R} = \omega_s v \qquad m = m_1 + m_2$$

$$\frac{v^2}{R} = \frac{Gm}{R^2} \implies \boxed{\omega_s^2 = \frac{Gm}{R^3}} \qquad (14)$$

The source of the GW emission is the total energy of the binary:

$$E_{kin} + E_{pot} = \frac{1}{2}\mu v^2 - \frac{Gm_1m_2}{R} \overset{v^2 = \frac{Gm}{R}}{=} -\frac{Gm_1m_2}{2R} = E_{orbit}$$

1) total energy must diminish due to GW emission, so R must diminish

2) if R diminishes, $\omega_s^2 = \frac{Gm}{R^3}$ grows

3) if ω_s grows, the emitted power grows as well (eq (13))

4) if the emitted power grows, R shrinks even more

$\Rightarrow$ THE RESULT OF THIS RUNAWAY PROCESS IS THE COALESCENCE OF THE BINARY SYSTEM.

The system can be analysed in the approximation that the orbit stays circular with a slowly varying radius $|\dot{R}| \ll v \iff \dot{\omega}_s \ll \omega_s^2$ (using $v = \omega_s R$ and (14))

The method with which one can account for the back-reaction of GW emission on the binary motion without going beyond linearised theory is to impose that the energy lost from the orbit is equal to the energy emitted in GW far away from the source:

$$\boxed{-\frac{dE_{orbit}}{dt} = [P]_{QUAD}}$$

Using (13), (14) and $E_{orbit} = -\frac{Gm_1m_2}{2R}$, from the above relation it is possible to get an equation that expresses how ω_s varies with time. It is customary to rewrite this in terms of the emitted GW frequency : $f_{GW} = \frac{\omega_{GW}}{2\pi} = \frac{\omega_s}{\pi}$

$$\boxed{\dot{f}_{GW} = \frac{96}{5}\pi^{8/3}\left(\frac{GM_c}{c^3}\right)^{5/3} f_{GW}^{11/3}} \qquad (15)$$

where one defines the $\boxed{\text{chirp mass}}$: $M_c = \mu^{3/5} m^{2/5}$

This equation is of fundamental importance since it allows to infer the chirp mass of the emitting system by observing the frequency and the frequency variation of the GW signal. It is the equation that the LIGO team has used to get the chirp mass of the systems they have observed. Note that, within linearised theory, all observable quantities depend solely on the specific combination of m_1 and m_2 which is the chirp mass. In order to infer the single masses of the two bodies composing the binary system one needs to resort to the full GR equations. Eq (15) is a differential equation that tells us how the GW frequency increases in time due to the emission of GWs and consequent shrinking of the orbit radius. If one attempts to integrate it, one encounters a divergence. In reality, this divergence is cut off by the last stages of the inspiral and the merger, that require full numerical GR to be described. One can write the solution to (15) in terms of the variable $\boxed{\tau = t - t_{coal}}$ where t_{coal} denotes the time at which one has the divergence:

$$f_{GW}(t) = \frac{1}{\pi}\left(\frac{5}{256}\right)^{3/8}\left(\frac{G M_c}{c^3}\right)^{-5/8}\left(\frac{1}{\tau}\right)^{3/8} \qquad (16)$$

Inserting numbers is the above formula one can realise that observing at low frequency allows one to see more cycles before coalescence:

<u>LIGO</u> : the first LIGO event had $M_c = 28\, M_\odot$ and started at $f_{GW} \simeq 35$ Hz : it lasted $\tau \simeq 0.2$ sec

LISA : a space-based interferometer can detect at much lower frequencies than LIGO : $10^{-4}\,Hz < f_{GW} < 10^{-2}\,Hz$

supposing as event a black hole binary of about $M_c \simeq 10^5 M_\odot$ the signal can last up to 3 years.

Let us now see how the amplitudes h_+ and $h_\times$ of the GW signal (12) are modified by accounting for back-reaction. Since the orbital radius and angular velocity of the "effective" particle now depend on time

$$x(t) = R(t)\cos\left(\frac{\phi(t)}{2}\right)$$

$$y(t) = R(t)\sin\left(\frac{\phi(t)}{2}\right)$$

$$\phi(t) = 2\int_{t_0}^{t} dt'\, \omega_s(t') \qquad (17)$$

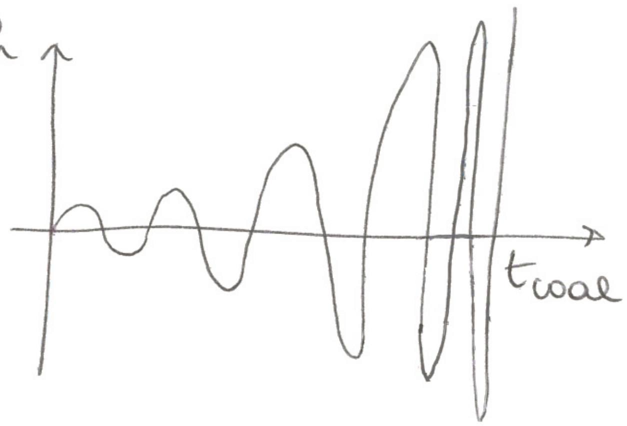
when one takes the second derivatives $\ddot{M}_{ij}$ one has in principle terms $\dot{R}(t)$ and $\dot{\omega}_s(t)$. It turns out however that these terms can be dropped, at least in first approximation, because the condition $\dot{\omega}_s \ll \omega_s^2$ [and therefore $|\dot{R}| \ll \omega_s R$] is always met within our linearized theory setting. From eq (15) one can in fact estimate the frequency (as a function of M_c) at which the above condition breaks down, and it turns out that it is always bigger than the typical frequency at which GR effects must be taken into account, i.e. the frequency corresponding to the innermost stable circular orbit. Therefore, as long as the linearised theory description is valid, one can always drop $\dot{\omega}_s$ and $\dot{R}$ and the GW amplitudes become simply

$$\boxed{\begin{aligned} h_+(t) &= \frac{4}{\pi}\left(\frac{G\mathcal{M}_c}{c^2}\right)^{\frac{5}{3}}\left(\frac{\pi f_{GW}(t_{ret})}{c}\right)^{\frac{2}{3}}\frac{1+\cos^2\theta}{2}\cos(\phi(t_{ret})) \\ h_\times(t) &= \frac{4}{\pi}\left(\frac{G\mathcal{M}_c}{c^2}\right)^{\frac{5}{3}}\left(\frac{\pi f_{GW}(t_{ret})}{c}\right)^{\frac{2}{3}}\cos\theta\,\sin(\phi(t_{ret})) \end{aligned}}$$

where we have rewritten them in terms of $\mathcal{M}_c$ and f_{GW}. The time dependent phase can be evaluated from (17), inserting (16) and integrating:

$$\phi(\tau) = -2\left(\frac{5G\mathcal{M}_c}{c^3}\right)^{-\frac{5}{8}}\tau^{5/8} + \phi_0$$

with ϕ_0 the phase at coalescence. From the above expressions and from (16) it appears that, as the system gets closer to merger $t \to t_{coal}$, τ decreases, the GW amplitude increases and the frequency as well. One recovers the so-called CHIRP SIGNAL:



Even if derived in the context of linearised theory, one recognizes the gross features of the LIGO detections: the chirping waveform and the behaviour with time to coalescence of the frequency as $f_{GW} \propto \tau^{-3/8}$.

Let us evaluate the amplitude of the GW signal measured by the LIGO detectors for the first event:

$$M_c = 28\ M_\odot$$

$$r \simeq d_L = 410\ \text{Mpc}$$

$$f_{GW} \simeq 35\ \text{Hz}$$

$$h \simeq \frac{4}{410\ \text{Mpc}} \left(\frac{G M_\odot}{c^2}\right)^{5/3} \left(\frac{28\ M_\odot}{M_\odot}\right)^{5/3} \left(\frac{\pi f_{GW}}{c}\right)^{2/3} \simeq 8 \cdot 10^{-22}$$

We find the number anticipated on page 29